Tiling tripartite graphs with 3-colorable graphs: The extreme case

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Abstract

There is a positive integer N_0 such that the following holds. Let $N \geq N_0$ such that N is divisible by h. If G is a tripartite graph with N vertices in each vertex class such that every vertex is adjacent to at least $2N/3 + 2h - 1$ vertices in each of the other classes, then G can be tiled perfectly by copies of $K_{h,h,h}$. This extends work by the authors [15] and also gives a sufficient condition for tiling by any fixed 3-colorable graph. Furthermore, we show that the minimum-degree $2N/3 + 2h - 1$ in our result can not be replaced by $2N/3 + h - 2$ and that if N is divisible by $6h$, then the required minimum degree is $2N/3 + h - 1$ for N large enough and this is tight.

1 Introduction

Here we extend the results of [15] on the tiling of 3-colorable graphs in tripartite graphs. Let H be a graph on h vertices, and let G be a graph on n vertices. An H-tiling of G is a subgraph of G which consists of vertex-disjoint copies of H and a *perfect H-tiling* of G is an H-tiling consisting of $\lfloor n/h \rfloor$ copies of H. In order to correspond with other results in this area, we call a perfect H-tiling an *H-factor*.

Hajnal and Szemerédi [7] settled the tiling problem for K_r by showing that each n-vertex graph G with $\delta(G) \geq (r-1)n/r$ contains a K_r-factor (it is easy to see that this is sharp). (Corrádi and Hajnal [4] proved the case $r = 3$.) Using Szemerédi's regularity lemma [17], Alon and Yuster [1, 2] obtained results on H-tiling for arbitrary H. Their results were later improved by various
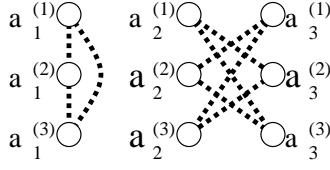
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Figure 1: The graph Γ_3, with the vertex classes in rows and the dotted lines representing **nonedges**. Vertex labels correspond to sets of vertices that occur when the graph is "blown up."

researchers [10, 9, 16, 11]. For further background in this field, we refer the reader to [15] but especially to the survey by Kühn and Osthus [12].

In this paper, we consider multipartite tiling, which restricts G to be an r-partite graph. For $r = 2$, this is an immediate consequence of the König-Hall Theorem (e.g. see [3]). Wang [18] considered $K_{s,s}$-factors in bipartite graphs for all $s > 1$, the second author [19] gave the best possible minimum degree condition for this problem.

As in [15], for a tripartite graph $G = (A, B, C; E)$, the graphs induced by (A, B), (A, C) and (B, C) are called the **natural bipartite subgraphs** of G. Let $\mathcal{G}_r(N)$ be the family of r-partite graphs with N vertices in each of its partition sets. In an r-partite graph G, $\bar{\delta}(G)$ stands for the minimum degree from a vertex in one partition set to any other partition set.

The graph Γ_3 in Figure 1 has $\bar{\delta}(\Gamma_3) = 2$ but no K_3-factor. In fact, the paper [13] showed that, if N is an odd multiple of 3, the so-called blow-up graph $\Gamma_3(N) \in \mathcal{G}_3(N)$ (where each edge of Γ_3 is replaced with a $K_{N/3,N/3}$ and each non-edge is replaced by an $(N/3) \times (N/3)$ bipartite graph with no edges) is the unique graph with $\bar{\delta} \geq 2N/3$ and no K_3-factor. As a result, this gives the following Corrádi-Hajnal-type theorem.

Theorem 1.1 ([13]) *If $G \in \mathcal{G}_3(N)$ satisfies $\bar{\delta}(G) \geq (2/3)N + 1$, then G contains a K_3-factor. Moreover, there is a $G \in \mathcal{G}_3(N)$ satisfying $\bar{\delta}(G) \geq (2/3)N - 1$ which has no K_3-factor.*

Theorem 1.2 (which appeared as Theorem 1.2 in [15]) shows that 2/3 is the correct coefficient of N required to have a $K_{h,h,h}$-factor.

Theorem 1.2 ([15]) *For any positive real number γ and any positive integer h, there is N_0 such that the following holds. Given an integer $N \geq N_0$ such that N is divisible by h, if G is a tripartite graph with N vertices in each vertex class such that ever vertex is adjacent to at least $(2/3 + \gamma)N$ vertices in each of the other classes, then G contains a $K_{h,h,h}$-factor.*

Our main result is the following more precise theorem.

Theorem 1.3 *Fix a positive integer h. Let $f(h)$ be the smallest value for which there exists an*

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N_0 such that if G is a balanced tripartite graph on $3N$ vertices, $N \geq N_0$, $h \mid N$, and each vertex is adjacent to at least $f(h)$ vertices in each of the other classes, then G contains a $K_{h,h,h}$-factor.

If $h \geq 2$ and $N = (6q + r)h$ with $0 \leq r < 6$, then

$$
\begin{array}{rcccl}
& f(h) & = & \frac{2N}{3} + h - 1, & \text{if } r = 0; \\
h\left\lceil \frac{2N}{3h} \right\rceil + h - 2 \leq & f(h) & \leq & h\left\lceil \frac{2N}{3h} \right\rceil + h - 1, & \text{if } r = 1, 2, 4, 5; \\
\frac{2N}{3} + h - 1 \leq & f(h) & \leq & \frac{2N}{3} + 2h - 1, & \text{if } r = 3.
\end{array}
$$

So, the result is tight for $N = 6h$, almost tight unless N/h is an odd multiple of 3 and, in the worst case, the upper and lower bounds differ by h.

Clearly the complete tripartite graph $K_{h,h,h}$ can itself be perfectly tiled by any 3-colorable graph on h vertices. Since $f(h) \leq \frac{2N}{3} + 2h - 1$ whenever N is divisible by h, we have the following corollary.

Corollary 1.4 *Let H be a 3-colorable graph of order h. There exists a positive integer N_0 such that if $N \geq N_0$ and N divisible by h, then every $G \in \mathcal{G}_3(N)$ with $\bar{\delta}(G) \geq \frac{2N}{3} + 2h - 1$ contains an H-factor.*

The lower bound is due to two constructions, one from [15], the other similar. They are stated in Proposition 1.5, and proven in Section 2.

Proposition 1.5 *Fix a positive integer h. There exists an N_0 such that*

(1). *if $h \geq 2$, $N \geq N_0$, $h \mid N$ and N/h is divisible by 3, then there is a graph $G_2 \in \mathcal{G}_3(N)$ with no $K_{h,h,h}$-factor and $\bar{\delta}(G_2) \geq 2N/3 + h - 2$;*

(2). *if $h \geq 3$, $N \geq N_0$, $h \mid N$ and N/h is not divisible by 3, then there is a graph $G_3 \in \mathcal{G}_3(N)$ with no $K_{h,h,h}$-factor and $\bar{\delta}(G_3) \geq h\left\lceil \frac{2N}{3h} \right\rceil + h - 3$; and*

As to the upper bound, we use Theorem 1.6 (Theorem 1.4 from [15]) to take care of the main case. Given $\gamma > 0$, we say that $G = \left(V^{(1)}, V^{(2)}, V^{(3)}; E\right) \in \mathcal{G}_3(N)$ is in *the extreme case with parameter γ* there are three sets A_1, A_2, A_3 such that $A_i \in V^{(i)}$, $|A_i| = \lfloor N/3 \rfloor$ for all i and

$$
d(A_i, A_j) := \frac{e(A_i, A_j)}{|A_i||A_j|} \leq \gamma
$$

for $i \neq j$.

Theorem 1.6 ([15]) *Given any positive integer h and any $\gamma > 0$, there exists an $\varepsilon > 0$ and an integer N_0 such that whenever $N \geq N_0$, and h divides N, the following occurs: If $G \in \mathcal{G}_3(N)$ satisfies $\bar{\delta}(G) \geq (2/3 - \varepsilon)N$, then either G contains a $K_{h,h,h}$-factor or G is in the extreme case with parameter γ.*

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Hence, for the upper bound, it is only necessary to assume that $G \in \mathcal{G}_3(N)$ is in the extreme case with parameter γ. The proof, given in Section 3, is detailed and involves a case analysis. Moreover, it requires the definition of a particular structure we call **the very extreme case**. Roughly, it means that the graph looks like $\Gamma_3(N)$ and the definition of the very extreme case is in Section 3.

Formally, the upper bound theorem is stated as follows:

Theorem 1.7 *Fix $h \geq 2$. There exists an N_0 such that if $N \geq N_0$ and $h \mid N$ then $G \in \mathcal{G}_3(N)$ and $\bar{\delta}(G) \geq h \left\lceil \frac{2N}{3h} \right\rceil + 2h - 1$ implies that G has a $K_{h,h,h}$-factor.*

Moreover, if $N \geq N_0$, $h \mid N$, $G \in \mathcal{G}_3(N)$ and $\bar{\delta}(G) \geq h \left\lceil \frac{2N}{3h} \right\rceil + h - 1$, then either G has a $K_{h,h,h}$-factor or G is in the very extreme case.

2 Lower bound

First, we cite a lemma (Lemma 2.1 in [15]) which permits sparse tripartite graphs with no triangles or quadrilaterals:

Lemma 2.1 *For each integer $d \geq 0$, there exists an n_0 such that, if $n \geq n_0$, there exists a balanced tripartite graph, $Q(n, d)$ on $3n$ vertices such that each of the 3 natural bipartite subgraphs are d-regular with no C_4 and $Q(n, d)$ has no K_3.*

Finally, we prove the lower bound itself.

Proof of Proposition 1.5.
Construction (1):
The construction in [15] satisfies the conditions we require.

Construction (2):
Let $h \geq 2$ and $N = (3q + r)h$ so that, in this case, $r \in \{1, 2\}$. Let G_3 be defined such that $V^{(i)} = A_1^{(i)} + A_2^{(i)} + A_3^{(i)}$ in which *column j* is defined to be the triple of the form $\left(A_j^{(1)}, A_j^{(2)}, A_j^{(3)} \right)$. Let the graph in column 1 be $Q(qh + rh - 1, rh + h - 4)$ if $rh + h - 4 \geq 0$ and empty otherwise, the graph in column 2 be $Q(qh, h - 3)$ and the graph in column 3 be $Q(qh + 1, h - 2)$. If two vertices are in different columns and different vertex-classes, then they are adjacent. It is easy to verify that $\bar{\delta}(G_3) = 2qh + rh + (h - 3) = h\lceil (2N)/(3h) \rceil + h - 3$. Suppose, by way of contradiction, that G_3 has a $K_{h,h,h}$-factor.

Since there are no triangles and no C_4's in any column, the intersection of a copy of $K_{h,h,h}$ with a column is either a star, with all leaves in the same vertex-class, or a set of vertices in the same vertex-class. So each copy of $K_{h,h,h}$ has at most $h + 1$ vertices in column 1, h vertices in column 2 and at most h vertices in column 3.

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There are three cases for a copy of $K_{h,h,h}$. Case 1 has h vertices in each column. Case 2 has $h+1$ vertices in column 1, $h-1$ vertices in column 2 and h vertices in column 3. Case 3 has $h+1$ vertices in column 1, h vertices in column 2 and $h-1$ vertices in column 3.

In Cases 1 and 2, having h vertices of a $K_{h,h,h}$ in column 3 implies that all of them are in the same vertex class. In Case 3, having h vertices in column 2 means that all are in the same vertex-class. Since $h+1$ vertices in column 1 means that they form a star, the remaining $h-1$ vertices in column 3 must be in the same vertex-class (the same vertex-class as the center of the star). Hence, every copy of $K_{h,h,h}$ has all of its column 3 vertices in the same vertex-class. Therefore, the number of copies of $K_{h,h,h}$ in a factor is at least $3\left\lceil \frac{qh+1}{h} \right\rceil = 3q + 3$, a contradiction because the factor has exactly $3q + r \leq 3q + 2$ copies of $K_{h,h,h}$. $\qquad\square$

3   The extreme case

Before we deal with the extreme case, we make the solution precise by describing a specific exclusionary case, which we deal with in Section 3.5.

Definition 3.1 *A balanced tripartite graph G on $3N$ vertices is in **the very extreme case** if the following occurs: First, there are integers N, q such that $N = (6q + 3)h$. Second, there are sets $A_j^{(i)} \subseteq V^{(i)}$ for $i, j \in \{1, 2, 3\}$, each with size at least $2qh + 1$, such that if $v \in A_j^{(i)}$ then v is nonadjacent to at most $3h - 3$ vertices in $A_{j'}^{(i')}$ whenever the pair $(A_j^{(i)}, A_{j'}^{(i')})$ corresponds to an edge in the graph Γ_3 with respect to the usual correspondence.*

The Main Theorem is proven by verifying the following:

Theorem 3.2 *Given any positive integer h, there exists a Δ, $0 < \Delta \ll h^{-1}$ and $N_0 = N_0(h)$ such that whenever $N \geq N_0$ and h divides N, the following occurs: If $G = \left(V^{(1)}, V^{(2)}, V^{(3)}; E\right)$ is a balanced tripartite graph on $3N$ vertices and G is in the extreme case with parameter Δ and $\bar{\delta}(G) \geq h\left\lceil \frac{2N}{3h} \right\rceil + h - 1$, then, either G has a $K_{h,h,h}$-factor or N is an odd multiple of $3h$ and G is in the very extreme case.*

If G is in the very extreme case, we can find the $K_{h,h,h}$-factor if $\bar{\delta}(G) \geq h\left\lceil \frac{2N}{3h} \right\rceil + 2h - 1$.

Throughout all of Section 3, assume that G is minimal, *i.e.*, no edge of G can be deleted so that the minimum degree condition still holds. We will have the usual sequence of constants:

$$\Delta \ll \Delta_1 \ll \Delta_2 \ll \Delta_3 \ll \Delta_4 \ll \Delta_5 \ll h^{-1}.$$

There are 4 parts to the proof. Part 1 begins with the extreme case and, if a $K_{h,h,h}$-factor is not found in G, then there is more of a structure to G, as described by Part 2. If a $K_{h,h,h}$-factor is not found in G when G is in Part 2, then it is either in Part 3a or Part 3b. Part 3a means that G is
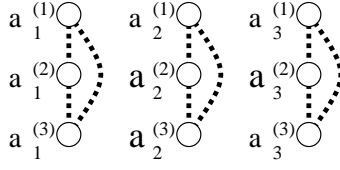
Figure 2: The graph $\Theta_{3\times3}$, with the vertex classes in rows and the dotted lines representing **nonedges**.

approximately $\Theta_{3\times3}(N/3)$. (See Figure 3.) In general, the graph $\Theta_{r\times n}$ is an r-partite graph with vertices $a_j^{(i)}$, $i = 1,\ldots,r$ and $j = 1,\ldots,n$ such that $a_j^{(i)} \sim a_{j'}^{(i')}$ if and only if $i \neq i'$ and $j \neq j'$. Part 3b means that G is approximately $\Gamma_3(N/3)$. (See Figure 1.)

We will assume for Parts 1, 2 and 3a (Sections 3.1, 3.2 and 3.3, respectively) that $\bar{\delta}(G) \geq h\left\lceil\frac{2N}{3h}\right\rceil + h - 1$. In Part 3b (Section 3.4), we will begin with the same assumption on $\bar{\delta}$, until we are left with the very extreme case. Then we will allow $\bar{\delta}(G) \geq h\left\lceil\frac{2N}{3h}\right\rceil + 2h - 1$ in Section 3.5 to complete the proof.

Definition 3.3 *For δ, $0 < \delta < 1$ a graph H and positive integer M, we say a graph G is δ-approximately $H(M)$ if $V(G)$ can be partitioned into $|V(H)|$ nearly-equally sized pieces, each of size M or $M + 1$, corresponding to a vertex of H so that for vertices $v, w \in V(H)$ with $v \not\sim_H w$, the parts of $V(G)$ corresponding to v and w have density between them less than δ.*

3.1 Part 1: The basic extreme case

For Part 1, we will prove that either a $K_{h,h,h}$-factor exists in G, or G is in Part 2.

Let $A^{(i)} \subset V^{(i)}$ for $i = 1, 2, 3$ be the three pairwise sparse sets given by the statement of the theorem and $B^{(i)} = V^{(i)} \setminus A^{(i)}$ for $i = 1, 2, 3$. We then define $\widetilde{A}^{(i)}$ to be the "typical" vertices with respect to $A^{(i)}$, $\widetilde{B}^{(i)}$ to be "typical" with respect to $B^{(i)}$, and $C^{(i)}$ are what remain. Formally, for $i = 1, 2, 3$,

$$
\begin{aligned}
\widetilde{A}^{(i)} &= \left\{ x \in V^{(i)} : \deg_{A^{(j)}}(x) \leq \Delta_1 |A^{(j)}|, \forall j \neq i \right\} \\
\widetilde{B}^{(i)} &= \left\{ y \in V^{(i)} : \deg_{A^{(j)}}(y) \geq (1 - \Delta_1)|A^{(j)}|, \forall j \neq i \right\} \\
C^{(i)} &= V^{(i)} \setminus \left(A^{(i)} \cup B^{(i)} \right)
\end{aligned}
$$

As a result, we have that $|B^{(i)} \setminus \widetilde{B}^{(i)}| \leq (2\Delta/\Delta_1)|B^{(i)}|$ and $|A^{(i)} \setminus \widetilde{A}^{(i)}| \leq (2\Delta/\Delta_1)|A^{(i)}|$. So, with $\Delta_1 = \Delta^{1/3}$, $|\widetilde{B}^{(i)}| \geq \left(1 - 2\Delta_1^2\right)|B^{(i)}| \geq \left(1 - 2\Delta_1^2\right)(2N/3)$ and $|\widetilde{A}^{(i)}| \geq \left(1 - 2\Delta_1^2\right) \geq |A^{(i)}| \geq \left(1 - 2\Delta_1^2\right)(N/3)$. We ignore round-off in computing sizes of $A^{(i)}$'s and $B^{(i)}$'s.

Step 1: There are large $\widetilde{A}^{(i)}$ sets

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Let $T = h \lfloor N/(3h) \rfloor$. We will eventually modify each of the sets $\widetilde{A}^{(i)}$ into sets $A_1^{(i)}$ that are either of size T or $T + h$. Let $N = (3q + r)h$ with $r \in \{0, 1, 2\}$. More precisely, the largest r sets $\widetilde{A}^{(i)}$ will be modified into sets $A_1^{(i)}$ of size $T + h$ and the smallest $3 - r$ sets $\widetilde{A}^{(j)}$ will be modified into sets $A_1^{(j)}$ of size T.

We will find, in $\widetilde{A}^{(1)} \cup \widetilde{A}^{(2)} \cup \widetilde{A}^{(3)}$, (vertex-disjoint) h-stars. We need the following lemma, proven in Section 3.6.

Lemma 3.4 *Let us be given $\epsilon > 0$ and a positive integer M.*

(1) *Let $(A^{(1)}, A^{(2)}; E)$ be a bipartite graph such that every vertex in $A^{(2)}$ is adjacent to at least d_1 vertices in $A^{(1)}$. Furthermore, $\left| |A^{(i)}| - M \right| < \epsilon M$ and $d_i < \epsilon M$ for $i = 1, 2$.*

Provided $\epsilon < ((h+1)h)^{-1}$, there is a family of vertex-disjoint copies of $K_{1,h}$ such that $\max\{0, d_1 - h + 1\}$ of them have centers in $A^{(1)}$.

(2) *Let $(A^{(1)}, A^{(2)}, A^{(3)}; E)$ be a tripartite graph such that every vertex not in $A^{(i)}$ is adjacent to at least d_i vertices in $A^{(i)}$, for $i = 1, 2, 3$. Furthermore, $\left| |A^{(i)}| - M \right| < \epsilon M$ and $d_i < \epsilon M$ for $i = 1, 2, 3$.*

Provided $\epsilon < (2(h+2)(h+1)h)^{-1}$, there is a family of vertex-disjoint copies of $K_{1,h}$ such that $\max\{0, d_i - h + 1\}$ of them have centers in $A^{(i)}$ and leaves in $A^{(i+1)}$ (index arithmetic is modulo 3).

With our degree condition, we can guarantee that each vertex not in $V^{(i)}$ is adjacent to at least $|\widetilde{A}^{(i)}| - T + h - 1$ vertices in $\widetilde{A}^{(i)}$. So, we use Lemma 3.4(2) with $d_i \geq |\widetilde{A}^{(i)}| - T + h - 1$ to construct the stars with the property that there are exactly enough centers in $\widetilde{A}^{(i)}$ such that, when removed, the resulting set has its size bounded above by either T or $T + h$, whichever is required. Place these centers into $Z^{(i)}$.

<center>**Step 2: There are small $\widetilde{A}^{(i)}$ sets**</center>

For a subgraph $K_{1,h,h}$, with $h \geq 2$, define the *center* to be the vertex that is adjacent to all others. We will also refer the remaining vertices as *leaves*, although their degree is $h + 1$.

We will find, in $B := \bigcup_{i=1}^{3} \left(\widetilde{B}^{(i)} \cup C^{(i)} \right)$, (vertex-disjoint) copies of $K_{1,h,h}$ such that $\max\{T + h - |\widetilde{A}^{(i)}|, 0\}$ copies having its center vertex in $B^{(i)}$ for the largest r sets $\widetilde{A}^{(i)}$ and such that $T - |\widetilde{A}^{(j)}|$ copies having the center vertex in $B^{(j)}$ for the smallest $3 - r$ sets $\widetilde{A}^{(j)}$. This will be accomplished with Lemma 3.5, proven in Section 3.6.

Lemma 3.5 *Given $\delta > 0$, there exists an $\epsilon = \epsilon(\delta) > 0$ such that the following occurs:*

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Let $(B^{(1)}, B^{(2)}, B^{(3)}; E)$ be a tripartite graph such that for all $i \neq j$, each vertex in $B^{(i)}$ is adjacent to at least $(1 - \epsilon)M$ vertices in $B^{(j)}$. Furthermore, $\left| |B^{(i)}| - 2M \right| < \epsilon M$.

If $(B^{(1)}, B^{(2)}, B^{(3)}; E)$ contains no copy of $K_{1,h,h}$ with 1 vertex in $B^{(1)}$, and h vertices in each of $B^{(2)}$ and $B^{(3)}$, then the graph $(B^{(1)}, B^{(2)}, B^{(3)}; E)$ is δ-approximately $\Theta_{3 \times 2}(M)$.

Lemma 3.5 can be repeatedly applied at most $\lceil \Delta_1(N/3) \rceil$ times, unless G is Δ_2-approximately $\Theta_{3 \times 3}(T)$. Here, we will want $\Delta_1 + 6\Delta_1^2 < \epsilon(\Delta_2)$. Add the center vertices of the $K_{1,h,h}$ subgraphs to the appropriate sets $\widetilde{A}^{(i)}$.

Place vertices from $C^{(i)}$ into the sets $\widetilde{A}^{(i)}$ so that $A_1^{(i)}$ is of size t or $t + h$, for $i = 1, 2, 3$ and that $\sum_{i=1}^{3} \left| A_1^{(i)} \right| = N$. Relabel the modified sets $\widetilde{A}^{(i)}$ with $A_1^{(i)}$.

Step 3: Finding a $K_{h,h}$-factor in B

Now we try to find a $K_{h,h}$-factor among the remaining vertices in B with the goal of matching them with the $A_1^{(i)}$ vertices. There are, however, some adjustments that should be made.

- Vertices which are in copies of $K_{1,h,h}$, where the center vertex is in some $A_1^{(i)}$, will be in a specified copy of $K_{h,h}$ in B.

- If $v \in Z^{(i)}$ is the center of a $K_{1,h}$ with leaves in $A_k^{(i)}$, then v will be assigned to $B^{(j)}$, where $\{j\} = \{1, 2, 3\} \setminus \{i, k\}$.

- Vertices $v \in C^{(i)}$ will be assigned to $B^{(j)}$ if v is adjacent to at least $(2\Delta_1)(N/3)$ vertices in $A_1^{(k)}$. Since $v \in C^{(i)}$ it will be assigned either to $B^{(j)}$ or to $B^{(k)}$, where $\{j, k\} = \{1, 2, 3\} \setminus \{i\}$.

This last statement results from the fact that if $v \in C^{(i)}$, then we may assume, without loss of generality, that v is adjacent to less than $(1 - 2\Delta_1^2)(2T)$ vertices in, say, $B^{(j)}$. Hence, v is adjacent to at least $(2\Delta_1^2)T$ vertices in $A^{(j)}$ and at least $(3\Delta_1/2)T$ vertices in $A_1^{(i)}$.

Moreover, we have that $|C^{(i)}| \leq 9\Delta_1^2 T$, $|Z^{(i)}| \leq 6\Delta_1^2 T$ and there are at most $4\Delta_1^2 T$ copies of $K_{1,h,h}$ with the center vertex in a given $A_1^{(i)}$.

Lemma 3.6 is proven in Section 3.6.

Lemma 3.6 Let us be given $\delta > 0$. Then there exists an $\epsilon = \epsilon(\delta) > 0$ and a positive integer $T_0 = T_0(\delta)$ such that the following occurs:

Let there be positive integers T_1, T_2, T_3 which are divisible by h and with $|T_i - T_j| \in \{0, h\}$, for all $i, j \in \{1, 2, 3\}$ and $T_1 > T_0$. Let $(B^{(1)}, B^{(2)}, B^{(3)}; E)$ be a tripartite graph such that for distinct indices $i, j, k \in \{1, 2, 3\}$, $|B^{(i)}| = T_j + T_k$. For all $i \neq j$, each vertex in $B^{(i)}$ is adjacent to at least $(1 - \epsilon)T_1$ vertices in $B^{(j)}$. We attempt to find a $K_{h,h}$-factor in the graph induced by $(B^{(1)}, B^{(2)}, B^{(3)}; E)$ with certain restrictions:
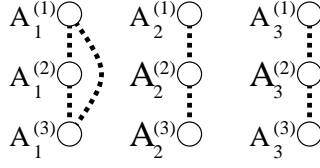
Figure 3: The graph that defines Part 2. A dotted line represents a sparse pair.

For each pair $(B^{(i)}, B^{(j)})$, there are at most ϵT_1 copies of $K_{h,h}$ which must be part of any factor. For each $B^{(i)}$, there are at most ϵT_1 vertices with the following property: v can only be in copies of $K_{h,h}$ in the pair $(B^{(i)}, B^{(j)})$ and v is adjacent to at least $(1 - \epsilon)T_1$ vertices in $B^{(i)}$.

If such a factor cannot be found, then, without loss of generality, the graph induced by $(B^{(1)}, B^{(2)}, B^{(3)}; E)$ can be partitioned such that $B^{(i)} = B^{(i)}[1] + B^{(i)}[2]$, $|B^{(i)}[1]| = T_1$ for $i = 1, 2, 3$ and $d\left(B^{(j)}[1], B^{(2)}[1]\right) \leq \delta$ and $d\left(B^{(j)}[2], B^{(2)}[2]\right) \leq \delta$ for $j = 1, 3$.

Then, match vertices in $C^{(i)}$ that are assigned to $B^{(j)}$ with h typical neighbors in $B^{(j)}[i]$ and those with $h - 1$ typical neighbors in $B^{(i)}[j]$. Finally, place the vertices that were moved into copies of $K_{h,h,h}$. All of these will be removed, allowing us to apply Lemma 3.6. If the appropriate $K_{h,h}$-factor cannot be found, then we are in the case of Part 2. The diagram that defines that case is in Figure 3.

Step 4: Completing the $K_{h,h,h}$-factor

We use Proposition 3.7, which allows us to complete a $K_{h,h}$-factor into a $K_{h,h,h}$-factor. The proof follows easily from König-Hall and is in Section 3.6.

Proposition 3.7 *Let $h \geq 1$.*

(1) *Let $G = (V^{(1)}, V^{(2)}; E)$ be a bipartite graph with $|V^{(1)}| = |V^{(2)}| = M$, h divides M, and each vertex is adjacent to at least $\left(1 - \frac{1}{2h^2}\right) M$ vertices in the other part. Then, we can find a $K_{h,h}$-factor in G.*

(2) *Let $G = (V^{(1)}, V^{(2)}, V^{(3)}; E)$ be a tripartite graph with $|V^{(1)}| = |V^{(2)}| = |V^{(3)}| = M$, h divides M, and each vertex is adjacent to at least $\left(1 - \frac{1}{4h^2}\right) M$ vertices in each of the other parts. Furthermore, let there be a $K_{h,h}$-factor in $(V^{(2)}, V^{(3)})$. Then, we can extend it into a $K_{h,h,h}$-factor in G.*

This allows us to find $K_{h,h,h}$-factors in each of $\left(A_1^{(1)}, B^{(2)}[3], B^{(3)}[2]\right)$, $\left(A_1^{(2)}, B^{(1)}[3], B^{(3)}[1]\right)$ and $\left(A_1^{(3)}, B^{(1)}[2], B^{(2)}[1]\right)$ which completes the $K_{h,h,h}$-factor in G.

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3.2 Part 2: G is approximately the graph in Figure 3

Remark. In this part, we must deal with the fact that the sets $A_2^{(2)}$ and $A_3^{(2)}$ may have close to the same number of vertices, but that number is not divisible by h. Much more work needs to be done in order to modify these sets so that their sizes become divisible by h. We think it is easier to see the basic arguments in the relatively shorter Part 1 before addressing the specific issues raised in Part 2.

Recall that each vertex is adjacent to at least $h\lceil\frac{2N}{3h}\rceil + h - 1$ vertices in each of the other pieces of the partition. Again, let $t = h\lfloor N/(3h)\rfloor$. We will transform the graph that is Δ_2-approximately a graph defined by Figure 3 with the vertices corresponding to sets of size $T \approx N/3$.

Before we begin, we must examine the behavior of $\left(A_2^{(1)} \cup A_3^{(1)}, A_2^{(3)} \cup A_3^{(3)}\right)$. If this is Δ_5- approximately $\Theta_{2\times 2}T$, then call the dense pairs $(E^{(1)}, E^{(3)})$ and $(F^{(1)}, F^{(3)})$. Otherwise, coincidence can only occur in either $V^{(1)}$ or $V^{(3)}$, but not both. Without loss of generality, we will assume that if there is such a coincidence, then it occurs in $V^{(1)}$.

We say that these pairs **coincide** with the sets $A_j^{(i)}$ if the typical vertices of, say $A_2^{(3)}$, have small intersection with those of $F^{(3)}$. We will determine the quantity that constitutes "small" later. If $(E^{(1)}, E^{(3)})$ and $(F^{(1)}, F^{(3)})$ both coincide with $(A_2^{(1)}, A_3^{(3)})$ and $(A_3^{(1)}, A_2^{(3)})$, then G is a graph that is approximately $\Theta_{3\times 3}(N/3)$ (Section 3.3). If $(E^{(1)}, E^{(3)})$ and $(F^{(1)}, F^{(3)})$ both coincide with $(A_2^{(1)}, A_2^{(3)})$ and $(A_3^{(1)}, A_3^{(3)})$, then approximately $\Gamma_3(N/3)$ (Section 3.4). Otherwise, coincidence can only occur in either $V^{(1)}$ or $V^{(3)}$, but not both. Without loss of generality, we will assume that if there is such a coincidence, then it occurs in $V^{(1)}$.

Let $V^{(i)} = A_1^{(i)} + A_2^{(i)} + A_3^{(i)} + C^{(i)}$, such that each $A_j^{(i)}$ has size between $\left(1 - 3\Delta_2^{2/3}\right)T$ and $\left(1 + 3\Delta_2^{2/3}\right)T$ and each vertex in $A_j^{(i)}$ is adjacent to at least θT vertices in each set $A_{j'}^{(i')}$ for which one of the following occurs:

- $i = 2$ and $j' \neq j$

- $i \in \{1, 3\}$, $j = 1$ and $j' \neq j$

- $i \in \{1, 3\}$, $i' = 2$ and $j \in \{2, 3\}$

- $i \in \{1, 3\}$, $i' = 4 - i$, $j \in \{2, 3\}$ and $j' = 1$

In other words, the vertices in $A_j^{(i)}$ are the ones that are typical according to the rules established by Figure 3. In addition, if, say $A_2^{(1)}$ coincides with $E^{(1)}$, then every vertex in $A_2^{(1)}$ is adjacent to at least θT vertices in $E^{(3)}$ and vice versa. If there is no coincidence, then let $E^{(1)}$ and $E^{(3)}$ be redefined so that every vertex in E_1 is adjacent to at least θT vertices in $E^{(3)}$ and vice versa. Similarly for $(F^{(1)}, F^{(3)})$.

Each vertex $c \in C^{(2)}$ has the property that, for all $j \in \{1, 2, 3\}$ and distinct $i', i'' \in \{1, 3\}$, if c is adjacent to fewer than $\Delta_3 T$ vertices in $A_j^{(i')}$, then c is adjacent to at least $\Delta_3 T$ vertices in $A_j^{(i'')}$.

Let $i \in \{1, 3\}$, each vertex $c \in C^{(i)}$ has the property that, for all $j \in \{1, 2, 3\}$, c cannot be adjacent to fewer than $\Delta_3 T$ vertices in either $A_2^{(2)}$ or $A_3^{(2)}$. Also, c cannot be adjacent to fewer than $\Delta_3 T$ vertices in both $A_1^{(2)}$ and $A_1^{(4-i)}$ or both $A_2^{(2)}$ and $F^{(4-i)}$ (if it exists) or both $A_3^{(2)}$ and $E^{(4-i)}$ (if it exists).

Trivially, each vertex in $V^{(i)}$ is adjacent to at least $(1/2 - \Delta_3)T$ vertices in at least two of $\{A_1^{(i')}, A_2^{(i')}, A_3^{(i')}\}$ and in at least two of $\{A_1^{(i'')}, A_2^{(i'')}, A_3^{(i'')}\}$, where i', i'' are distinct members of $\{1, 2, 3\} \setminus \{i\}$. This is particularly important for vertices in $C^{(i)}$.

Step 1: Ensuring small $A_j^{(i)}$ sets

First, take each triple $\left(A_j^{(1)}, A_j^{(2)}, A_j^{(3)}\right)$, $j = 1, 2, 3$, and construct disjoint copies of stars so that there are at most t non-center vertices in each set $A_j^{(i)}$. As in Part 1, we use the fact that every vertex is adjacent to at least $h \lceil \frac{2N}{3h} \rceil + h - 1$ vertices in each of the other parts as well as Lemma 3.4. For $i, j = 1, 2, 3$, place $|A_j^{(i)}| - T$ centers from $A_j^{(i)}$ into a set $Z^{(i)}$.

Step 2: Fixing the size of $A_j^{(i)}$ sets

We have sets $A_j^{(i)}$ which have $|A_j^{(i)}| \leq T$ and the remaining vertices are in sets $C^{(i)} \cup Z^{(i)}$. Since N is divisible by h, we can place the vertices $C^{(i)} \cup Z^{(i)}$ arbitrarily into sets $A_1^{(i)}$, $A_2^{(i)}$ and $A_3^{(i)}$ so that the resulting sets $A_j^{(i)}$ have cardinality t or $t + h$ and for $j = 1, 2, 3$,

$$|A_j^{(1)}| + |A_j^{(2)}| + |A_j^{(3)}| = N.$$

For this purpose, if $N/h \cong 1 \pmod 3$, add h vertices to each of $A_2^{(1)}$, $A_1^{(2)}$ and $A_3^{(3)}$. If $N/h \cong 2 \pmod 3$, add h vertices to all sets $A_j^{(i)}$, except $A_2^{(1)}$, $A_1^{(2)}$ and $A_3^{(3)}$.

Step 3: Partitioning the sets

We will partition each set $A_j^{(i)}$ into two pieces, as close as possible to equal size, but which have size divisible by h. This must have the property that a typical vertex in $A_j^{(i)}$ has at least $\left(1 - 2\Delta_4 - 6\Delta_2^{2/3}\right)(T/2)$ neighbors in each piece of the partition of $A_{j'}^{(i')}$, $i' \neq i$, $j' \neq j$. Moreover, if a vertex has degree at least $\Delta_3 T$ in a set, it has degree at least $(\Delta_3/3)(T/2)$ in each of the two partitions. Such a partition exists, almost surely, provided N is large enough, if the partition is random.

Assign to each part a permutation, $\sigma \in \Sigma_3$, which assigns $j = \sigma(i)$. (Σ_3 denotes the symmetric group that permutes the elements of $\{1, 2, 3\}$.) Each part assigned to σ will be the same size.

Step 4: Assigning vertices

The former $C^{(i)}$ vertices, as well as star-leaves and star-centers, may only be able to form a $K_{h,h,h}$ with respect to one particular permutation.

For example, consider a vertex c which had been in $C^{(1)}$ but is now in $A_1^{(1)}$. Then, for either the pair $(A_2^{(2)}, A_3^{(3)})$ or the pair $(A_3^{(2)}, A_2^{(3)})$, the vertex c is adjacent to at least $(1/2 - \Delta_3)T$ in one set and at least $\Delta_3 T$ vertices in the other; otherwise, it would have been a typical vertex in $A_1^{(1)}$, $A_2^{(1)}$ or $A_3^{(1)}$.

Assume that c is adjacent to at least $\Delta_3 T$ vertices in $A_3^{(2)}$ and at least $(1/2 - \Delta_3)T$ vertices in $A_2^{(3)}$. In this case, if c were placed into the partition corresponding to the identity permutation, then exchange c with a typical vertex in the partition assigned to (23), using cycle notation of permutations.

In a similar fashion, if there is a star with center in, say $A_2^{(1)}$, and leaves in, say $A_1^{(2)}$, then we will use it to form a $K_{h,h,h}$ with respect to the permutation $(12) \in \Sigma_3$. Again, if any such leaf or center was in the wrong partition, exchange it with a typical vertex in the other partition.

The number of leaves in any set is at most $2h\left(6\Delta_2^{2/3}T + h\right)$ and the number of centers is at most $2\left(6\Delta_2^{2/3}t + h\right)$, the number of $C^{(i)}$ vertices is at most $9\Delta_2^{2/3}T$. So, if N is large enough, the total number of typical vertices in any $A_j^{(i)}$ which were exchanged is at most $2(12h+21)\Delta_2^{2/3}T+4h^2+4h$. With the partition established and the $C^{(i)}$, $Z^{(i)}$ and leaf vertices in the proper part, we consider the triple formed by three sets:

- $A_1^{(2)}$, which will also be denoted $\widetilde{S}^{(2)}$

- the union of the piece of $A_2^{(1)}$ corresponding to (12) and the piece of $A_3^{(1)}$ corresponding to (132), denoted $\widetilde{S}^{(1)}$, and

- the union of the piece of $A_2^{(3)}$ corresponding to (132) and the piece of $A_3^{(3)}$ corresponding to (12), denoted $\widetilde{S}^{(3)}$.

Let the graph induced by the triple $\left(\widetilde{S}^{(1)}, \widetilde{S}^{(2)}, \widetilde{S}^{(3)}\right)$ be denoted $\widetilde{\mathcal{S}}$.

Step 5: Finding a $K_{h,h,h}$ cover in $\widetilde{\mathcal{S}}$

Let $T_0 = |A_1^{(2)}|$. First, take each $K_{1,h}$ in $\mathcal{S}'$ and complete it to form disjoint copies of $K_{h,h,h}$, using unexchanged typical vertices. This can be done if Δ_4 is small enough. Remove all such $K_{h,h,h}$'s containing stars.

Second, take each c which had been a member of some $C^{(i)}$ and use it to complete a $K_{h,h,h}$. We can guarantee, because of the random partitioning, that c is adjacent to at least $(\Delta_3/3)T_0$ vertices in one set and $(1/3 - 2\Delta_3)T_0$ vertices in the other. Without loss of generality, let $c \in \widetilde{S}^{(1)}$ with degree at least $(\Delta_3/3)T_0$ in $\widetilde{S}^{(2)}$ and at least $(1/4 - 2\Delta_3)T_0$ in $\widetilde{S}^{(3)}$. Since $\Delta_3 \gg \Delta_2$, we can guarantee h

neighbors of c in $\widetilde{S}^{(2)}$ among unexchanged typical vertices and, if $\Delta_3 \ll \Delta_4 \ll 1$, then h common neighbors of those among unexchanged typical vertices in $N(c) \cap \widetilde{S}^{(3)}$. Finally, $\Delta_4 \ll h^{-1}$ implies this $K_{h,h}$ has at least $h-1$ more common neighbors in $\widetilde{S}^{(1)}$. This is our $K_{h,h,h}$ and we can remove it. Do this for all former members of a $C^{(i)}$.

Third, take each exchanged typical vertex and put it into a $K_{h,h,h}$ and remove it. Throughout this process, we have removed at most $C_h \sqrt{\Delta_2} \times T_0$ vertices where C_h is a constant depending only on h. What remains are three sets of the same size, $T' \geq (1 - C_h \sqrt{\Delta_2}) T_0$, with each vertex in $\widetilde{S}^{(1)}$ adjacent to at least, say $(1/2 - 2\Delta_4) T'$, vertices in $\widetilde{S}^{(3)}$ and vice versa. Each vertex in $\widetilde{S}^{(1)}$ and in $\widetilde{S}^{(3)}$ is adjacent to at least $(1/2 - 2\Delta_4) T'$ vertices in $\widetilde{S}^{(2)}$ and each vertex in $\widetilde{S}^{(2)}$ is adjacent to at least $(1/2 - 2\Delta_4) T'$ vertices in $\widetilde{S}^{(1)}$ and in $\widetilde{S}^{(3)}$.

Lemma 3.8, from [19], shows that we can find a factor of $\left(\widetilde{S}^{(1)}, \widetilde{S}^{(3)}\right)$ with vertex-disjoint copies of $K_{h,h}$ unless $\left(\widetilde{S}^{(1)}, \widetilde{S}^{(3)}\right)$ is approximately $\Theta_{2\times2}(T/2)$. In that case, find the factor and finish to form a factor of $\widetilde{S}$ of vertex-disjoint copies of $K_{h,h,h}$ via König-Hall.

Lemma 3.8 (Z. [19]) *For every $\epsilon > 0$ and integer $h \geq 1$, there exists an $\alpha > 0$ and an N_0 such that the following holds. Suppose that $N > N_0$ is divisible by h. Then every bipartite graph $G = (A, B; E)$ with $|A| = |B| = N$ and $\delta(G) \geq (1/2 - \alpha)N$ either contains a $K_{h,h}$-factor, or contains $A' \subseteq A$, $B' \subseteq B$ such that $|A'| = |B'| = N/2$ and $d(A, B) \leq \epsilon$.*

Lemma 3.9 states, in particular, that if a random partition results in $\left(\widetilde{S}^{(1)}, \widetilde{S}^{(3)}\right)$ being approximately $\Theta_{2\times2}(T/2)$ with high probability, then $\left(A_2^{(1)} \cup A_3^{(1)}, A_2^{(3)} \cup A_3^{(3)}\right)$ is approximately $\Theta_{2\times2}(T)$. The proof of Lemma 3.9 follows from similar arguments to those in the proof of Lemma 3.3 of [13] and in Section 3.3.1 of [14] so we omit it.

Lemma 3.9 *For every $\epsilon > 0$ and integer $h \geq 1$, there exists a $\beta > 0$ and positive integer T_0 such that if $T \geq T_0$ the following holds. Let (A, B) be a bipartite graph such that $|A|, |B| \in \{2T - h, 2T, 2T + h\}$ with minimum degree at least $(1 - \epsilon)t$ and is minimal with respect to this condition. Let $A' \subset A$, $B' \subset B$, $|A'| = |B'| = T$ be chosen uniformly at random. If*

$$\Pr\{(A', B') \text{ contains a subpair with density at most } \epsilon\} \geq 1/4$$

then (A, B) is β-approximately $\Theta_{2\times2}(T)$.

We can, therefore, assume the existence of $(E^{(1)}, E^{(3)})$ and $(F^{(1)}, F^{(3)})$. Otherwise, Lemmas 3.8 and 3.9 imply that $\widetilde{S}$ has a $K_{h,h,h}$-factor.

As a result, recall that we let the typical vertices in the dense pairs in $\left(A_2^{(1)} \cup A_3^{(1)}, A_2^{(3)} \cup A_3^{(3)}\right)$ be denoted $(E^{(1)}, E^{(3)})$ and $(F^{(1)}, F^{(3)})$. If the dense pairs do not coincide, then we will work to ensure that $|E^{(1)} \cap \widetilde{S}^{(1)}| = |E^{(3)} \cap S_3'|$ and $|F^{(1)} \cap \widetilde{S}^{(1)}| = |F^{(3)} \cap \widetilde{S}^{(3)}|$ and both are divisible by

h. Do this by moving vertices from $\left(A_2^{(1)} \cap E^{(1)}\right) \setminus \widetilde{S}^{(1)}$ into $\left(A_2^{(1)} \cap E^{(1)}\right) \cap \widetilde{S}^{(1)}$ and move the same number from $\left(A_2^{(1)} \cap F^{(1)}\right) \cap \widetilde{S}^{(1)}$ into $\left(A_2^{(1)} \cap F^{(1)}\right) \setminus \widetilde{S}^{(1)}$. In addition, move vertices from $\left(A_2^{(3)} \cap E^{(3)}\right) \setminus \widetilde{S}^{(3)}$ into $\left(A_2^{(3)} \cap E^{(3)}\right) \cap \widetilde{S}^{(3)}$ and move the same number from $\left(A_2^{(3)} \cap F^{(3)}\right) \cap \widetilde{S}^{(3)}$ into $\left(A_2^{(3)} \cap F^{(3)}\right) \setminus \widetilde{S}^{(3)}$.

This can be done unless one of the intersections $A_j^{(i)} \cap E^{(i)}$ or $A_j^{(i)} \cap F^{(i)}$ is too small. This implies the coincidence that we discussed at the beginning of this part. But then, we have guaranteed that the remaining vertices of $A_2^{(1)}$ are not only typical in that set but also typical in $E^{(1)}$. The same is true of $A_3^{(1)}$ and $F^{(1)}$.

Now, we want to move vertices in $V^{(3)}$ to ensure that $|E^{(3)} \cap \widetilde{S}^{(3)}| = |A_2^{(1)} \cap \widetilde{S}^{(1)}|$ and $|F^{(3)} \cap \widetilde{S}^{(3)}| = |A_3^{(1)} \cap \widetilde{S}^{(1)}|$. Note that we have ensured that both $|A_2^{(1)} \cap \widetilde{S}^{(1)}|$ and $|A_3^{(1)} \cap \widetilde{S}^{(1)}|$ are divisible by h and approximately $T/2$.

We can do this as follows: Move vertices from $E^{(3)} \cap A_2^{(3)} \setminus \widetilde{S}^{(3)}$ to $(E^{(3)} \cap A_2^{(3)}) \cap \widetilde{S}^{(3)}$ and move the same amount from $(F^{(3)} \cap A_2^{(3)}) \cap \widetilde{S}^{(3)}$ to $(F^{(3)} \cap A_2^{(3)}) \setminus \widetilde{S}^{(3)}$. Also move vertices from $(E^{(3)} \cap A_3^{(3)}) \setminus \widetilde{S}^{(3)}$ to $(E^{(3)} \cap A_3^{(3)}) \cap \widetilde{S}^{(3)}$ and move the same amount from $(F^{(3)} \cap A_3^{(3)}) \cap \widetilde{S}^{(3)}$ to $(F^{(3)} \cap A_3^{(3)}) \setminus \widetilde{S}^{(3)}$. Since none of the intersections are small, this is possible. Complete this to vertex-disjoint copies of $K_{h,h,h}$ in $\widetilde{S}$ by Proposition 3.7.

Step 6: Completing the $K_{h,h,h}$-factor in G

Now that we have found a $K_{h,h,h}$ that corresponds permutations (12) and (132), we consider permutations in Σ_3. For a $\sigma \in \Sigma_3 \setminus \{(12),(132)\}$, let $\mathcal{S}(\sigma) \overset{\text{def}}{=} \left(S_{\sigma(1)}^{(1)}, S_{\sigma(2)}^{(2)}, S_{\sigma(3)}^{(3)}\right)$ be a triple of parts formed by the random partitioning after the exchange of vertices has taken place. The set $S_{\sigma(i)}^{(i)}$ is a subset of $A_{\sigma(i)}^{(i)}$. We have also ensured that $s_\sigma \overset{\text{def}}{=} |S_{1,\sigma(1)}| = |S_{2,\sigma(2)}| = |S_{3,\sigma(3)}|$ and s_σ is divisible by h. It is now easy to ensure that this triple contains a $K_{h,h,h}$-factor:

First, take each star in $\mathcal{S}(\sigma)$ and complete it to form disjoint copies of $K_{h,h,h}$, using unexchanged typical vertices. This can be done if Δ_4 is small enough. Remove all such $K_{h,h,h}$'s containing stars.

Second, take each c which had been a member of some $C^{(i)}$ and use it to complete a $K_{h,h,h}$. We can guarantee, because of the random partitioning, that c is adjacent to at least $(\Delta_3/3)s_\sigma$ vertices in one set and $(2/3 - 2\Delta_3)s_\sigma$ vertices in the other. Without loss of generality, let $c \in S_{\sigma(1)}^{(1)}$ with degree at least $(\Delta_3/3)s_\sigma$ in $S_{\sigma(2)}^{(2)}$ and at least $(1/2 - 2\Delta_3)s_\sigma$ in $S_{\sigma(3)}^{(3)}$. Since $\Delta_3 \gg \Delta_2$, we can guarantee h neighbors of c in $S_{\sigma(2)}^{(2)}$ among unexchanged typical vertices and, if $\Delta_3 \ll \Delta_4 \ll 1$, then h common neighbors of those among unexchanged typical vertices in $N(c) \cap S_{\sigma(3)}^{(3)}$. Finally, $\Delta_4 \ll h^{-1}$ implies this $K_{h,h}$ has at least $h-1$ more common neighbors in $S_{\sigma(1)}^{(1)}$. This is our $K_{h,h,h}$ and we can remove it. Do this for all former members of a $C^{(i)}$.

Finally, take each exchanged typical vertex and put it into a $K_{h,h,h}$ and remove it. Throughout

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this process, we have removed at most $C_h\sqrt{\Delta_2}\times s_\sigma$ vertices where C_h is a constant depending only on h. What remains are three sets of the same size, $s' \geq (1 - C_h\sqrt{\Delta_2})s_\sigma$, with each vertex adjacent to at least, say $(1 - 2\Delta_4)\,s'$, vertices in each of the other parts. If N is large enough, then we can use the Blow-up Lemma or Proposition 3.7(2) to complete the factor of $\mathcal{S}(\sigma)$ by copies of $K_{h,h,h}$.

3.3 Part 3a: G is approximately $\Theta_{3\times 3}\left(\lfloor N/3\rfloor\right)$

Figure 3 shows $\Theta_{3\times 3}$ and we are in the case where G is approximately $\Theta_{3\times 3}(\lfloor N/3\rfloor)$, where $a_j^{(i)}$ and $a_j^{(i')}$ being connected with a dotted line means that the pair $\left(A_j^{(i)}, A_{j'}^{(i')}\right)$ is sparse.

We will assume for this part that each vertex is adjacent to at least $h\left\lceil\frac{2N}{3h}\right\rceil + h - 1$ vertices in each of the other pieces of the partition. Again, let $t = h\lfloor N/(3h)\rfloor$.

We will transform the Δ_2-approximately $\Theta_{3\times 3}\left(\lfloor N/3\rfloor\right)$ by partitioning $V^{(i)}$, $i = 1, 2, 3$, into four sets, as follows: $V^{(i)} = A_1^{(i)} + A_2^{(i)} + A_3^{(i)} + C^{(i)}$, such that each $A_j^{(i)}$ has size between $(1 - \sqrt{\Delta_2})t$ and $(1 + \sqrt{\Delta_2})t$ and each vertex in $A_j^{(i)}$ is adjacent to at least θt vertices in each set $A_{j'}^{(i')}$ where $i' \neq i$ and $j' \neq j$.

Each vertex $c \in C^{(i)}$ has the property that, for all $j \in \{1, 2, 3\}$ and distinct $i', i'' \in \{1, 2, 3\} \setminus \{i\}$, if c is adjacent to fewer than $\Delta_3 t$ vertices in $A_j^{(i')}$, then c is adjacent to at least $\Delta_3 t$ vertices in $A_j^{(i'')}$; otherwise c is in some set $A_j^{(i)}$. Furthermore, c is adjacent to at least $(1/2 - \Delta_3)t$ vertices in at least two of $\left\{A_1^{(i')}, A_2^{(i')}, A_3^{(i')}\right\}$ and in at least two of $\left\{A_1^{(i'')}, A_2^{(i'')}, A_3^{(i'')}\right\}$.

Step 1: Ensuring small $A_j^{(i)}$ sets

First, take each triple $\left(A_j^{(1)}, A_j^{(2)}, A_j^{(3)}\right)$, $j = 1, 2, 3$, and construct disjoint copies of stars so that there are at most t non-center vertices in each set $A_j^{(i)}$. We use the fact that every vertex is adjacent to at least $h\left\lceil\frac{2N}{3h}\right\rceil + h - 1$ vertices in each of the other parts as well as Lemma 3.4. For $i, j = 1, 2, 3$, place $|A_j^{(i)}| - t$ centers from $A_j^{(i)}$ into a set $Z^{(i)}$.

Step 2: Fixing the size of $A_j^{(i)}$ sets

We have sets $A_j^{(i)}$ which have $|A_j^{(i)}| \leq t$ and the remaining vertices are in sets $C^{(i)} \cup Z^{(i)}$. Since N is divisible by h, we can place the vertices $C^{(i)} \cup Z^{(i)}$ arbitrarily into sets $A_1^{(i)}$, $A_2^{(i)}$ and $A_3^{(i)}$ so that the resulting sets $A_j^{(i)}$ have cardinality t or $t + h$ and for $j = 1, 2, 3$,

$$|A_j^{(1)}| + |A_j^{(2)}| + |A_j^{(3)}| = N.$$

For this purpose, we could place these vertices first to ensure that all $|A_j^{(i)}|$ become of size exactly t. If $N = 3th + h$ then, for $i = 1, 2, 3$, add all of the remaining $C^{(i)} \cup Z^{(i)}$ to $A_i^{(i)}$. If $N = 3th + 2h$ then, for $i = 1, 2, 3$, add all of the remaining $C^{(i)} \cup Z^{(i)}$ to $A_j^{(i)}$, $j \neq i$.

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Step 3: Partitioning the sets

We will randomly partition each set $A_j^{(i)}$ into two pieces, as close as possible to equal size but which have size divisible by h, and assign them to a permutation, $\sigma \in \Sigma_3$, which assigns $j = \sigma(i)$. (Σ_3 denotes the symmetric group that permutes the elements of $\{1, 2, 3\}$.) Each part assigned to σ will be the same size. We call a vertex in $A_j^{(i)}$ a **typical vertex** if it was not in $C^{(i)}$ and is neither a star-leaf nor a star-center.

Note that a typical vertex in $A_j^{(i)}$ has at least $(1 - 2\Delta_4 - 2\sqrt{\Delta_2})(t/2)$ neighbors in each piece of the partition of $A_{j'}^{(i')}$, $i' \neq i$, $j' \neq j$, almost surely – provided N is large enough and the partition was as equitable as possible. Moreover, if a vertex has degree at least $\Delta_3 t$ in a set, it has degree at least $(\Delta_3/3)(t/2)$ in each of the two partitions.

Step 4: Assigning vertices

The former $C^{(i)}$ vertices, as well as star-leaves and star-centers may only be able to form a $K_{h,h,h}$ with respect to one particular permutation.

For example, consider a vertex c which had been in $C^{(1)}$ but is now in $A_1^{(1)}$. Then, for either the pair $(A_2^{(2)}, A_3^{(3)})$ or the pair $(A_3^{(2)}, A_2^{(3)})$, the vertex c is adjacent to at least $(1/2 - \Delta_3)t$ in one set and at least $\Delta_3 t$ vertices in the other. It is easy to see that, since $\Delta_2 \ll \Delta_3$, that if this were not true, then it would have been possible to place c into one of the sets $A_1^{(1)}$, $A_2^{(1)}$ or $A_3^{(1)}$.

Assume that c is adjacent to at least $\Delta_3 t$ vertices in $A_3^{(2)}$ and at least $(1/2 - \Delta_3)t$ vertices in $A_2^{(3)}$. In this case, if c were placed into the partition corresponding to the identity permutation, then exchange c with a typical vertex in the partition assigned to (23), using cycle notation of permutations.

In a similar fashion, if there is a star with center in, say $A_2^{(1)}$, and leaves in, say $A_1^{(2)}$, then we will form a $K_{h,h,h}$ with respect to the permutation $(12) \in \Sigma_3$. Again, if any such leaf or center was in the wrong partition, exchange it with a typical vertex in the other partition.

The number of leaves in any set is at most $2h(\sqrt{\Delta_2}t + h)$ and the number of centers is at most $2(\sqrt{\Delta_2}t + h)$, the number of $C^{(i)}$ vertices is at most $3\sqrt{\Delta_2}t$. So, if N is large enough, the total number of typical vertices in any $A_j^{(i)}$ which were exchanged is at most $(2h + 6)\sqrt{\Delta_2}t$.

Step 5: Completing the cover

For some $\sigma \in \Sigma_3$, let $\mathcal{S}(\sigma) \stackrel{\text{def}}{=} \left(S_{\sigma(1)}^{(1)}, S_{\sigma(2)}^{(2)}, S_{\sigma(3)}^{(3)}\right)$ be a triple of parts formed by the random partitioning after the exchange has taken place. The set $S_{\sigma(i)}^{(i)}$ is a subset of $A_{\sigma(i)}^{(i)}$. We have also ensured in Step 3 that $s_\sigma \stackrel{\text{def}}{=} |S_{\sigma(1)}^{(1)}| = |S_{\sigma(2)}^{(2)}| = |S_{\sigma(3)}^{(3)}|$ and s_σ is divisible by h. It is now easy to ensure that this triple contains a $K_{h,h,h}$-factor:

First, take each star in $\mathcal{S}(\sigma)$ and complete it to form disjoint copies of $K_{h,h,h}$, using unexchanged typical vertices. This can be done if Δ_4 is small enough. Remove all such $K_{h,h,h}$'s containing stars.

Second, take each c which had been a member of some $C^{(i)}$ and use it to complete a $K_{h,h,h}$. We can guarantee, because of the random partitioning, that c is adjacent to at least $(\Delta_3/3)s_\sigma$ vertices in one set and $(2/3 - 2\Delta_3)s_\sigma$ vertices in the other. Without loss of generality, let $c \in S_{\sigma(1)}^{(1)}$ with degree at least $(\Delta_3/3)s_\sigma$ in $S_{\sigma(2)}^{(2)}$ and at least $(1/2 - 2\Delta_3)s_\sigma$ in $S_{\sigma(3)}^{(3)}$. Since $\Delta_3 \gg \Delta_2$, we can guarantee h neighbors of c in $S_{\sigma(2)}^{(2)}$ among unexchanged typical vertices and, since $\Delta_3 \ll \Delta_4 \ll 1$, h common neighbors of those among unexchanged typical vertices in $N(c) \cap S_{\sigma(3)}^{(3)}$. Finally, $\Delta_4 \ll h^{-1}$ implies this $K_{h,h}$ has at least $h-1$ more common neighbors in $S_{\sigma(1)}^{(1)}$. This is our $K_{h,h,h}$ and we can remove it. Do this for all former members of a $C^{(i)}$.

Finally, take each exchanged typical vertex and put it into a $K_{h,h,h}$ and remove it. Throughout this process, we have removed at most $\Delta_2^{1/3} s_\sigma$ vertices if Δ_2 is small enough. What remains are three sets of the same size, $s' \geq \left(1 - \Delta_2^{1/3}\right) s_\sigma$, with each vertex adjacent to at least, say $(1 - 2\Delta_4) s'$, vertices in each of the other parts. If N is large enough, then we can use the Blow-up Lemma or Proposition 3.7(2) to complete the factor of $\mathcal{S}(\sigma)$ by copies of $K_{h,h,h}$.

3.4 Part 3b: G is approximately $\Gamma_3\left(\lfloor N/3\rfloor\right)$

Figure 1 shows Γ_3 and we are in the case where G is approximately $\Gamma_3(\lfloor N/3\rfloor)$, where $a_j^{(i)}$ and $a_j^{(i')}$ being connected with a dotted line means that the pair $\left(A_j^{(i)}, A_{j'}^{(i')}\right)$ is sparse.

We will assume for this part that each vertex is adjacent to at least $h\left\lceil\frac{2N}{3h}\right\rceil + h - 1$ vertices in each of the other pieces of the partition. We also assume that G is not in the very extreme case. We must deal with the very extreme case separately.

Let $t \overset{\text{def}}{=} h\lfloor N/(3h)\rfloor$. We will transform the Δ_2-approximately $\Gamma_3\left(\lfloor N/3\rfloor\right)$ by partitioning $V^{(i)}$, $i = 1, 2, 3$, into four sets, as follows: $V^{(i)} = A_1^{(i)} + A_2^{(i)} + A_3^{(i)} + C^{(i)}$, such that each $A_j^{(i)}$ has size between $(1 - \sqrt{\Delta_2})t$ and $(1 + \sqrt{\Delta_2})t$ and each vertex in $A_1^{(i)}$ is adjacent to at least $(1 - \Delta_3)t$ vertices in each set $A_{j'}^{(i')}$ where $i' \neq i$ and $j' \in \{2, 3\}$. For $j = 2, 3$, $A_j^{(i)}$ is adjacent to at least $(1 - \Delta_3)t$ vertices in each set $A_1^{(i')}$ and $A_j^{(i')}$, where $i' \neq i$.

Each vertex $c \in C^{(i)}$ has the property that, for all $j \in \{1, 2, 3\}$ and distinct $i', i'' \in \{1, 2, 3\} \setminus \{i\}$, if c is adjacent to fewer than $\Delta_3 t$ vertices in $A_j^{(i')}$, then c is adjacent to at least $\Delta_3 t$ vertices in $A_j^{(i'')}$. Furthermore, c is adjacent to at least $(1/2 - \Delta_4)t$ vertices in at least two of $\left\{A_1^{(i')}, A_2^{(i')}, A_3^{(i')}\right\}$ and $\left\{A_1^{(i'')}, A_2^{(i'')}, A_3^{(i'')}\right\}$.

Without loss of generality, we will assume that both $|A_2^{(1)}| \geq |A_3^{(1)}|$ and $|A_2^{(2)}| \geq |A_3^{(2)}|$.

Step 1: Ensuring small $A_j^{(i)}$ sets

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In each set $V^{(i)}$, we construct a set $Z^{(i)} = Z^{(i)}[1] + Z^{(i)}[2] + Z^{(i)}[3]$ that will contain star-centers.

If $|A_2^{(3)}| > |A_3^{(3)}|$, then $A_2^{(i)}$ is larger than $A_3^{(i)}$ for $i = 1, 2, 3$. Use Lemma 3.4(1) to construct $\max\left\{\min\{|A_2^{(i)}| - t, t - |A_3^{(i)}|\}, 0\right\}$ disjoint copies of $K_{1,h}$ in the pair[1] $(A_2^{(i)}, A_3^{(i+1)})$ with centers in $A_2^{(i)}$. Place these centers into $Z^{(i)}[3]$.

If $|A_2^{(3)}| < |A_3^{(3)}|$, we do something similar except that first we use Lemma 3.4(1) to create the appropriate number of stars in $(A_2^{(1)}, A_3^{(2)})$ and $(A_2^{(2)}, A_3^{(1)})$ with the centers in $A_2^{(1)}$ and $A_2^{(2)}$, respectively. Place these centers into $Z^{(1)}[3]$ and $Z^{(2)}[3]$, respectively. Then, we apply Lemma 3.4(1) to the pair $(A_3^{(3)}, A_2^{(2)})$. (This $A_2^{(2)}$ is the possibly modified set, with star-centers removed.)

By the conditions on Lemma 3.4(1), we see that each remaining set $A_j^{(i)}$ is of size at most t. Now, apply Lemma 3.4(2) to the triple $\left(A_1^{(1)}, A_1^{(2)}, A_1^{(3)}\right)$. For star-centers in $A_1^{(i)}$, place $t - |A_2^{(i)}|$ into $Z^{(i)}[2]$ and $t - |A_3^{(i)}|$ into $Z^{(i)}[3]$.

Step 2: Fixing the size of the $A_j^{(i)}$ sets for $j = 1, 2, 3$

We now attempt to "fill up" the sets $A_j^{(i)}$. Let $s_{i,j}$ be the targeted size. There are several cases according to the divisibility of N/h. Let $N/h = 6q + r$ where $0 \leq r < 6$.

- $r = 0$: $s_{i,j} = t$ for $i = 1, 2, 3$ and $j = 1, 2, 3$

- $r = 1$: $s_{i,j} = t$ for $i = 1, 2, 3$ and $j = 1, 3$; and $s_{i,2} = t + h$ for $i = 1, 2, 3$

- $r = 2$: $s_{i,1} = t$ for $i = 1, 2, 3$; and $s_{i,j} = t + h$ for $i = 1, 2, 3$ and $j = 2, 3$

- $r = 3$: $s_{i,j} = t$ for $i = 1, 2, 3$ and $j = 1, 2, 3$

- $r = 4$: $s_{i,1} = t$ for $i = 1, 2, 3$; and $s_{1,3} = s_{2,3} = s_{3,2} = t$; and $s_{1,2} = s_{2,2} = s_{3,3} = t + h$

- $r = 5$: $s_{i,1} = t$ for $i = 1, 2, 3$; and $s_{i,j} = t + h$ for $i = 1, 2, 3$ and $j = 2, 3$

The cases of $r = 0, 3$ and $r = 2, 5$ are diagrammed in Figure 4 and the cases of $r = 1$ and $r = 4$ are diagrammed in Figure 5.

Place vertices of $Z^{(i)}[j]$ into $A_j^{(i)}$ for $i = 1, 2, 3$ and $j = 1, 2, 3$. Furthermore, place vertices from $C^{(i)}$ into $A_j^{(i)}$ for $i = 1, 2, 3$ and $j = 1, 2, 3$, ensuring that we still have the case that $|A_j^{(i)}| \leq s_{i,j}$.

As usual, we call a vertex in $A_j^{(i)}$ a typical vertex if it was neither in $C^{(i)}$ nor is either a star-leaf or a star-center. For $j = 2, 3$, let $\mathcal{A}_j = \left(A_j^{(1)}, A_j^{(2)}, A_j^{(3)}\right)$. We remove some copies of $K_{h,h,h}$ from among typical vertices of these sets as follows:

- $r = 1$: One from $\mathcal{A}_2$.

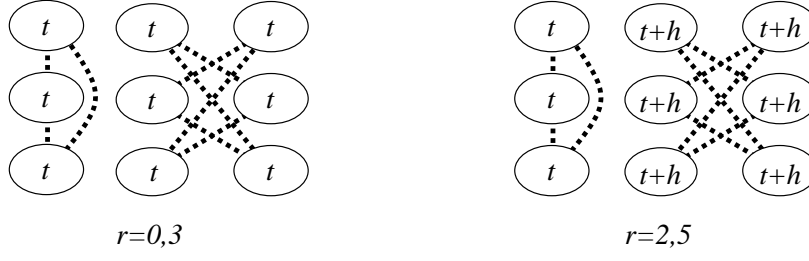[1]Arithmetic in the indices is always done modulo 3.

Figure 4: $N = (6q + r)h$ with $r = 0, 3$ and $r = 2, 5$, respectively. $t = 2qh + h\lfloor r/3 \rfloor$
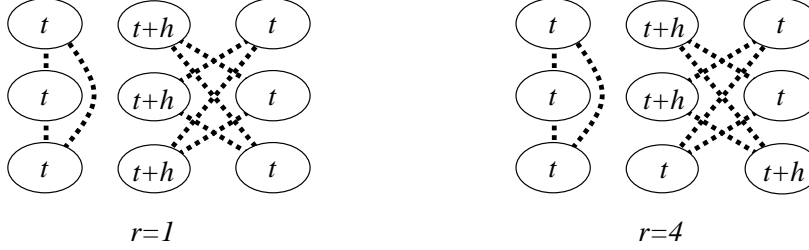

Figure 5: $N = (6q + r)h$ with $r = 1$ and $r = 4$, respectively. $t = 2qh + h\lfloor r/3 \rfloor$

- $r = 2$: One from each of $\mathcal{A}_2$ and $\mathcal{A}_3$.

- $r = 4$: One from $\mathcal{A}_2$.

- $r = 5$: Two from $\mathcal{A}_2$.

Recalling $N = (6q + r)h$, each set is of size $2qh$ or $2qh + h$. Here we note that $t_f \overset{\text{def}}{=} h\lfloor t/(2h) \rfloor = qh$. Also, $t_c \overset{\text{def}}{=} h\lceil t/(2h) \rceil = qh$ if $r = 0, 1, 2$ and $t_c = (q+1)h$ if $r = 3, 4, 5$.

Step 3a: Partitioning the sets ($r \neq 3$)

Let $r \in \{0, 1, 2, 4, 5\}$. Partition each $A_1^{(i)}$ set into parts of nearly equal size. Each part of the partition will receive a label $\sigma \in \{1, 2, 3\} \times \{2, 3\}$. Now, partition each $A_j^{(i)}$ as follows:

Each $A_1^{(i)}$ will be split into two pieces: one of size t_f and another of size t_c. Unless both $r = 4$ and $i = 3$, assign the smaller one with label $(i, 2)$ and the larger with label $(i, 3)$. If they are the same size, then assign them arbitrarily. If $r = 4$ and $i = 3$, then assign the one of size t_f with label $(3, 3)$ and the one of size t_c with $(3, 2)$.

Each $A_2^{(i)}$ will be split into two pieces. Unless both $r = 4$ and $i \in \{1, 2\}$, both pieces will be of size t_f and will be assigned $(i', 2)$ and $(i'', 3)$ arbitrarily, where $\{i, i', i''\} = \{1, 2, 3\}$.
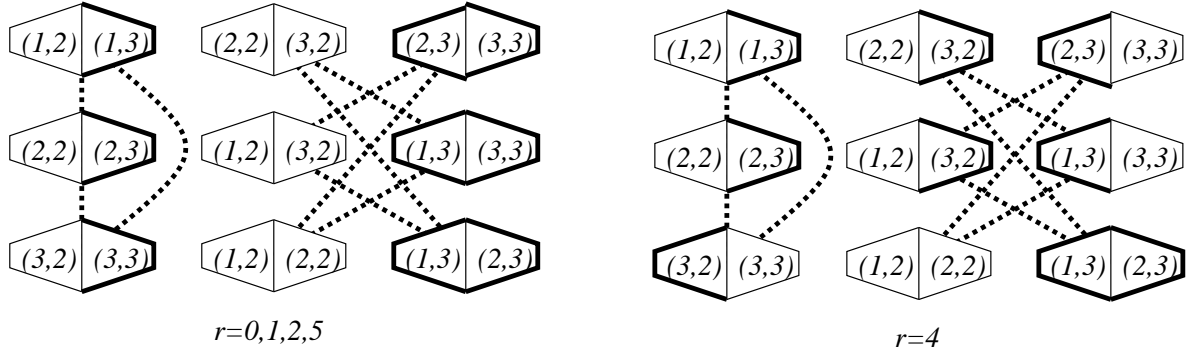
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Figure 6: Partitioning the sets. The light outlined half of a set is the piece of size t_f, the bold outlined half of a set is the piece of size t_c.

> If $r = 4$ and $i \in \{1, 2\}$, the one of size t_f is labeled $(3, 2)$ and the one of size t_c, is labeled $(3 - i, 2)$.
>
> Each $A_3^{(i)}$ will be split into two pieces. Unless both $r = 4$ and $i \in \{1, 2\}$, both pieces will be of size t_c and will be assigned $(i', 2)$ and $(i'', 3)$ arbitrarily, where $\{i, i', i''\} = \{1, 2, 3\}$. If $r = 4$ and $i \in \{1, 2\}$, the one of size t_f is labeled $(3, 3)$ and one of size t_c is labeled $(5 - i, 3)$.

Figure 6 diagrams the partitioning.

Partitioning the sets at random again ensures that the above can be accomplished so that all of the vertices' neighborhoods maintain roughly the same proportion, as in Part 3a, Step 3.

Now we proceed to Step 4.

Step 3b: Partitioning the vertices ($r = 3$, not the very extreme case)

Let $r = 3$ (recall $N = (6q + r)h$) and let G not be in the very extreme case. It may be possible that there are additional stars $K_{1,h}$ between sparse pairs. If it is possible to create enough such stars so as to move star-centers into $Z^{(i)}$, then we can have at least one of these sets $A_j^{(i)}$ of size at most $2qh$. If we are not able to do this, G must be in the the very extreme case. Without loss of generality, the set to be made small is either $A_1^{(1)}$ or $A_3^{(1)}$.

- Suppose vertices are removed to make $|A_1^{(1)}| = 2qh$. We will make the set $A_2^{(1)}$ of size $(2q+2)h$ by adding vertices from the sets $C^{(1)}$, $Z^{(1)}[2]$ and $Z^{(1)}[1]$.

- Suppose vertices are removed to make $|A_3^{(1)}| = 2qh$. We will make the set $A_2^{(1)}$ of size $(2q+2)h$ by adding vertices from the sets $C^{(1)}$, $Z^{(1)}[3]$ and $Z^{(1)}[1]$.

In each case, if the vertices in $Z^{(1)}[1]$ that were placed into $A_2^{(1)}$ were themselves originally in $A_2^{(1)}$, then we just treat them as typical vertices again, ignoring the star that was formed. Note that all sets are of size $(2q+1)h$, except $|A_2^{(1)}| = (2q+2)h$ and either $A_1^{(1)}$ or $A_3^{(1)}$, which has size $2qh$. If $A_1^{(1)}$ is the small set, then remove one copy of $K_{h,h,h}$ in the triple $\left(A_3^{(1)}, A_3^{(2)}, A_3^{(3)}\right)$.

Now we partition each set as follows: Each $A_1^{(i)}$ will have one piece of size qh with label $(1,3)$. The other set will have label $(1,2)$ size $(q+1)h$ in the case of $A_1^{(2)}$ and $A_1^{(3)}$ and either qh or $(q+1)h$ in the case of $A_1^{(1)}$. The set $A_2^{(1)}$ is partitioned into two pieces of size $(q+1)h$, one labeled $(2,2)$, the other labeled $(3,2)$. For $A_2^{(i)}$, $i = 2,3$, we have one piece of size qh and labeled $(1,2)$ and the other of size $(q+1)h$, labeled $(5-i,2)$. For $A_3^{(1)}$, it will have two pieces of size qh, one labeled $(2,3)$, the other $(3,3)$. Finally, for $A_3^{(i)}$, $i = 2,3$, we have one piece of size qh with label $(5-i,3)$ and the other will have size either qh or $(q+1)h$ and label $(1,3)$.

Partitioning the sets at random again ensures that the above can be accomplished so that all of the vertices' neighborhoods maintain roughly the same proportion, as in Part 3a, Step 3.

Now, we can proceed to Step 4.

Step 4: Assigning vertices

For any $\sigma \in \{1,2,3\} \times \{2,3\}$, we will show that the $Z^{(i)}$ and $C^{(i)}$ vertices, in any $A_j^{(i)}$ can be assigned to one of the two parts of the partition.

For example, consider a vertex c which had been in $C^{(1)}$ but is now in $A_1^{(1)}$. Then, for either the pair $(A_2^{(2)}, A_2^{(3)})$ or the pair $(A_3^{(2)}, A_3^{(3)})$, the vertex c is adjacent to at least $(1/2 - \delta)t$ in one set and at least $\Delta_3 t$ vertices in the other. If such a pair is $(A_2^{(2)}, A_2^{(3)})$ then if c were labeled $(1,2)$ exchange it with a typical vertex with label $(1,3)$.

Now, for example, consider a vertex c which had been in $C^{(1)}$ but is now in $A_2^{(1)}$. It is easy to check that for either the pair $(A_1^{(2)}, A_2^{(3)})$ or the pair $(A_1^{(3)}, A_2^{(2)})$, the vertex c is adjacent to at least $(1/2 - \Delta_3)t$ in one set and at least $\Delta_3 t$ vertices in the other. If such a pair is, say, $(A_1^{(2)}, A_2^{(3)})$, and c is not labeled $(2,2)$, then exchange it for a typical vertex of that label.

Without loss of generality, this takes care of those vertices $c \in C^{(i)}$.

Now we consider stars. All star-centers are in sets $A_2^{(i)}$ or $A_3^{(i)}$. Without loss of generality, assume z is such a center in $A_2^{(1)}$ and the leaves are in $V^{(2)}$. If the leaves are in $A_1^{(2)}$, then z must have been a member of $A_1^{(1)}$ originally. So, z and its leaves must have label $(2,2)$. If the leaves are in $A_2^{(2)}$, then z must have been a member of $A_3^{(1)}$ originally. So, z and its leaves must have label $(3,2)$. Exchange z with typical vertices to ensure this.

Finally, we consider typical vertices moved from $A_2^{(i)} \cup A_3^{(i)}$ to $A_1^{(i)}$. Without loss of generality, suppose z is such a vertex in $A_1^{(1)}$. If z were originally from $A_2^{(1)}$, then it is a typical vertex with respect to $A_2^{(2)}$ and $A_2^{(3)}$ and z should receive label $(1,2)$. Otherwise, it is typical with respect to $A_3^{(2)}$ and $A_3^{(3)}$ and z should receive label $(1,3)$.

This completes the verification that all moved vertices can receive at least one label of the $A_j^{(i)}$ set in which it is placed.

Step 5: Completing the cover

For any $\sigma \in \{1, 2, 3\} \times \{2, 3\}$, let $\mathcal{S}(\sigma)$ be one of the triples defined above. We can finish as in Part 3a, Step 5.

3.5   The very extreme case

Recall the very extreme case:

> There are integers N, q such that $N = (6q + 3)h$. There are sets $A_j^{(i)}$ for $i, j \in \{1, 2, 3\}$, with sizes at least $2qh + 1$, such that if $v \in A_j^{(i)}$ then v is nonadjacent to at most $3h - 3$ vertices in $A_{j'}^{(i')}$ whenever the pair $(A_j^{(i)}, A_{j'}^{(i')})$ corresponds to an edge in the graph Γ_3 with respect to the usual correspondence.

In this case, we must raise the minimum degree condition to $2N/3 + 2h - 1$. Recalling Part 4, Step 3b, we were able to proceed if we were able to make one of the sets $A_j^{(i)}$ small by means of creating stars. Each vertex in $A_2^{(2)}$ is adjacent to at least $|A_{1,3}| - N/3 + 2h - 1$ vertices in $A_3^{(1)}$. Using Lemma 3.4(1), we have that there is a family of $|A_3^{(1)}| - N/3 + h$ vertex-disjoint stars with centers in $A_3^{(1)}$. We move the centers to $A_2^{(1)}$. Then we can proceed from Part 4, Step 4.

3.6   Proofs of Lemmas

Proof of Lemma 3.4.

(1) Let $\delta_1 = d_1 - h + 1$. If the stars cannot be created greedily, then there is a set $S \subset A^{(1)}$ and $T \subset A^{(2)}$ such that $|S| \leq \delta_1 - 1$ and $|T| = |S|h$ and each vertex in $A^{(1)} \setminus S$ is adjacent to less than $h - 1$ vertices in $A^{(2)} \setminus T$. In this case,

$$(d_1 - |S|)|A^{(2)} \setminus T| \leq e(A^{(1)} \setminus S, A^{(2)} \setminus T) \leq (h - 1)|A^{(1)} \setminus S|.$$

This gives

$$
\begin{aligned}
|S| &\geq \delta_1 - (h - 1)\frac{|A^{(1)} \setminus S| - |A^{(2)} \setminus T|}{|A^{(2)} \setminus T|} \\
&\geq \delta_1 - (h - 1)\frac{|A^{(1)}| - |A^{(2)}| + (h - 1)|S|}{|A^{(2)}| - h|S|} \\
&\geq \delta_1 - (h - 1)\frac{(h + 1)\epsilon M}{(1 - (h + 1)\epsilon)M}.
\end{aligned}
$$

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If $\epsilon < (h^2 + h)^{-1}$, then this gives $|S| > \delta_1 - 1$. Since $|S|$ is an integer, $|S| \geq \delta_1$, contradicting the condition we put on $|S|$.

(2) Let $\delta_i = \max\{0, d_i - h + 1\}$ for $i = 1, 2, 3$. If, say, $\delta_3 = 0$, then apply part (1) to the pair $(A^{(2)}, A^{(3)})$ to create δ_2 vertex-disjoint stars with centers in $A^{(2)}$. Let $Z^{(2)}$ be the set of the centers. Apply part (1) to $(A^{(1)}, A^{(2)} \setminus Z^{(2)})$ and we can find δ_1 vertex-disjoint stars with centers in $A^{(1)}$ if $2\epsilon < (h^2 + h)^{-1}$.

So, we may assume that $\delta_i > 0$ for $i = 1, 2, 3$. Note that if it is possible to construct $\delta_1 + \delta_2$ disjoint copies of $K_{1,h}$ in $(A^{(1)}, A^{(2)})$ with centers, $Z^{(1)} \subset A^{(1)}$, then we can finish with application of part (1). To see this, apply part (1) to $(A^{(3)}, A^{(1)} \setminus Z^{(1)})$, with $3\epsilon < (h^2 + h)^{-1}$, creating δ_3 stars with centers $Z^{(3)} \in A^{(3)}$. Then apply part (1) to $(A^{(2)}, A^{(3)} \setminus Z^{(3)})$ ($2\epsilon < (h^2 + h)^{-1}$). There will be δ_1 stars remaining in $(A^{(1)}, A^{(2)})$ which are vertex-disjoint from the rest.

So, we will assume that it is not possible to create $\delta_1 + \delta_2$ vertex-disjoint copies of $K_{1,h}$ in $(A^{(1)}, A^{(2)})$ with centers in $A^{(1)}$. That means there is an $S \subset A^{(1)}$ and a $T \subset A^{(2)}$ such that $|S| < \delta_1 + \delta_2$, $|T| = h|S|$ and every vertex in $A^{(1)} \setminus S$ is adjacent to at most $h - 1$ vertices in $A^{(2)} \setminus T$.

Now apply part (1) to $(A^{(3)}, A^{(1)} \setminus S)$ to obtain δ_3 vertex-disjoint copies of $K_{1,h}$ with centers $Z^{(3)} \subset A^{(3)}$. (Here, we need $3\epsilon < (h^2 + h)^{-1}$.) Next, apply part (1) to $(A^{(2)}, A^{(3)} \setminus Z^{(3)})$ to obtain δ_2 vertex-disjoint copies of $K_{1,h}$ with centers $Z^{(2)} \setminus A^{(2)}$. (Here, we need $2\epsilon < (h^2 + h)^{-1}$.) Finally, apply part (1) to $\left(A^{(1)}, A^{(2)} \setminus (Z^{(2)} \cup T)\right)$ to obtain δ_1 vertex-disjoint copies of $K_{1,h}$ with centers $Z^{(1)} \subset A^{(1)}$. (Here, we need $(2h+2)\epsilon < (h^2 + h)^{-1}$.) But, because no vertex in $A^{(1)} \setminus S$ is adjacent to h vertices in $A^{(2)} \setminus (Z^{(2)} \cup T)$, it must be the case that $Z^{(1)} \subset S$ and our $\delta_1 + \delta_2 + \delta_3$ copies of $K_{1,h}$ are, indeed, vertex-disjoint.

$\square$

Proof of Lemma 3.5. We can first apply the following theorem of Erdős, Frankl and Rödl [6]:

Theorem 3.10 *For every $\epsilon' > 0$ and graph F, there is a constant n_0 such that for any graph G of order $n \geq n_0$, if G does not contain F as a subgraph, then G contains a set E' of at most $\epsilon' n^2$ edges such that $G \setminus E'$ contains no K_r with $r = \chi(F)$.*

Here, $F = K_{1,h,h}$ and $r = 3$.

So, after removing at most $\epsilon'(3M)^2$ edges, we have that the number of vertices in each part that are adjacent to at least $\sqrt{\epsilon}M$ vertices in each of the other two parts is at least $\left(1 - \frac{18\epsilon'}{\sqrt{\epsilon} - \epsilon}\right) M$.

So, now $2\left(1 - \frac{9\epsilon'}{\sqrt{\epsilon} - \epsilon}\right) M \leq |B^{(i)}| \leq 2\left(1 + \frac{\epsilon}{2}\right) M$ and each vertex is adjacent to at least $\left(1 - \frac{18\epsilon'}{\sqrt{\epsilon} - \epsilon}\right) M$ vertices in each of the other two parts.

Finally, we use a version of a proposition appearing in [13], rephrased below:

Proposition 3.11 *For a Δ small enough, there exists $\epsilon'' > 0$ such that if H is a tripartite graph with at least $2(1 - \epsilon'')t$ vertices in each vertex class and each vertex is nonadjacent to at most $(1 + \epsilon'')t$ vertices in each of the other classes. Furthermore, let H contain no triangles. Then, each vertex class is of size at most $2(1 + \epsilon'')t$ and H is Δ-approximately $\Theta_{3\times 2}(t)$.*

By guaranteeing $\epsilon'' \gg \epsilon' \gg \epsilon$ and $\delta = \Delta(\epsilon'') + \epsilon''$, the lemma follows. $\square$

Proof of Lemma 3.6.

Let ϵ' be chosen such that $\epsilon' \ll \delta$.

For this lemma, we partition the possibilities according to whether the pairs $(B^{(i)}, B^{(j)})$ are approximately $\Theta_{2\times 2}(t_1)$. That is, there are two sets of size t_1 which have density less than ϵ'. Minimality gives the rest.

In addition, we say that graphs $\Theta_{2\times 2}(t_1)$ **coincide** if there are sets $\widetilde{B}^{(i)} \subseteq B^{(i)}$, $\widetilde{B}^{(j)} \subseteq B^{(j)}$, $\widetilde{B}^{(k)} \subseteq B^{(k)}$, all of size t_1, such that both $(\widetilde{B}^{(i)}, \widetilde{B}^{(j)})$ and $(\widetilde{B}^{(j)}, \widetilde{B}^{(k)})$ have density less than ϵ'.

Case 1: No pair is $\Theta_{2\times 2}(t_1)$

For each distinct $i, j, k \in \{1, 2, 3\}$, partition $B^{(i)}$ into two pieces, $B^{(i)}[j]$ and $B^{(i)}[k]$ with $|B^{(i)}[j]| = t_j$ and $|B^{(i)}[k]| = t_k$. If this partition is done uniformly at random, then with probability approaching 1, each vertex in $B^{(i)}[k]$ is adjacent to at least $(1/2 - \epsilon^{1/2})t_k$ vertices in $B^{(j)}[k]$. So there exists a partition such that each vertex in $B^{(i)}$ is adjacent to at least $(1/2 - \epsilon^{1/2})t_1$ vertices in each of the pieces $B^{(j)}[k]$, $j, k \neq i$ and such that the pair $(B^{(2)}[1], B^{(3)}[1])$ fails to contain a subpair with $\lfloor t_1/2 \rfloor$ vertices in each part and density at most $\epsilon^{1/3}$.

The vertices that are reserved will have to be placed in the proper set. For example, if a reserved $K_{h,h}$ is in the pair $(B^{(i)}, B^{(j)})$, then those vertices will need to be in the pair $(B^{(i)}[k], B^{(j)}[k])$. So, we exchange vertices in $B^{(i)}[k]$ for vertices in $B^{(i)}[j]$ so that reserved vertices are in the proper place. At most $4(\epsilon + \epsilon)t_1$ vertices are either reserved or moved in each set $B^{(i)}[j]$. After such exchanges occur, place the moved vertices into vertex-disjoint copies of $K_{h,h}$ that lie entirely within the given pairs. This can be done because each vertex not in $B^{(i)}$ is adjacent to almost half of the vertices in both $B^{(i)}[j]$ and $B^{(i)}[k]$.

Consider what remains of these sets. The number of vertices is still divisible by h and at most $8h(\epsilon)t_1$ have been placed into these copies of $K_{h,h}$. We look for a perfect $K_{h,h}$-factor in each of the pairs $(B^{(1)}[3], B^{(2)}[3])$, $(B^{(1)}[2], B^{(3)}[2])$ and $(B^{(2)}[1], B^{(3)}[1])$. Recall that each of these pairs has minimum degree at least $(1/2 - \epsilon^{1/2})t_1$. Utilizing a lemma in [19] – stated as Lemma 3.8 in section 3.2 – we are able to find such a factor unless at least one of those pairs is $\alpha(\epsilon^{1/2})$-approximately $\Theta_{2\times 2}(t_1/2)$. (Minimality gives the other sparse pair.)

Lemma 3.9 says that if random selections give a graph that is approximately $\Theta_{2\times 2}(t_1/2)$, then the original graph was, too. So, along with Lemma 3.8, it establishes that if, after moving our vertices,

we are unable to complete our $K_{h,h}$-cover in $(B_i(k), B_j(k))$ with nontrivial probability, then the pair (B_i, B_j) is ϵ'-approximately $\Theta_{2\times2}(t_1)$, where $\epsilon' = \beta(\alpha(\epsilon^{1/2}))$.

Since none of the pairs is ϵ'-approximately $\Theta_{2\times2}(t_1)$, we can find the required factor of $\left(B^{(1)}, B^{(2)}, B^{(3)}\right)$ by copies of $K_{h,h}$.

Case 2: Exactly one pair is $\Theta_{2\times2}(t_1)$

Here, we will assume that $B^{(1)} = \widetilde{B}^{(1)} + \widehat{B}^{(1)}$ and $B^{(2)} = \widetilde{B}^{(2)} + \widehat{B}^{(2)}$, where $|\widetilde{B}^{(1)}| = |\widehat{B}^{(2)}| = t_1$ and $d(\widetilde{B}^{(1)}, \widehat{B}^{(2)}), d(\widehat{B}^{(1)}, \widetilde{B}^{(2)}) \leq \epsilon'$. A random partition of $B^{(1)}$ into pieces, with probability approaching 1 as t_1 approaches infinity, will partition $\widetilde{B}^{(1)}$ into two approximately equal pieces. In particular, let the **typical vertices** in $\widetilde{B}^{(1)}$ be those that are nonadjacent to at most $(\epsilon')^{1/2}t_1$ in $\widehat{B}^{(2)}$. There are at most $(\epsilon')^{1/2}t_1$ such vertices. A similar conclusion can be drawn from $\widetilde{B}^{(2)}$, $\widehat{B}^{(1)}$ and $\widehat{B}^{(2)}$.

In this case, we randomly partition $B^{(1)}$, $B^{(2)}$ and $B^{(3)}$ into the sets $B^{(i)}[k]$ as prescribed. Exchange the vertices as we have done above and complete both the reserved and exchanged vertices to form copies of $K_{h,h}$. This encompasses at most $8h\epsilon t_1$ vertices. Exchange vertices in $B^{(1)}[3]$ with vertices in $B^{(1)}[2]$ and vertices in $B^{(2)}[3]$ with vertices in $B^{(2)}[1]$ so that there are exactly $h\lfloor t_1/(2h)\rfloor$ typical vertices of $\widetilde{B}^{(1)}$ in $B^{(1)}[3]$ and $h\lfloor t_1/(2h)\rfloor$ typical vertices of $\widehat{B}^{(2)}$ in $B^{(2)}[3]$. Let the rest of the vertices, not matched into a $K_{h,h}$, in $B^{(1)}[3]$ be typical vertices in $\widehat{B}^{(1)}$ and the rest of the vertices in $B^{(2)}[3]$ be typical in $\widetilde{B}^{(2)}$. Using Proposition 3.7(1) on each pair of sets of typical vertices in $(B^{(1)}[3], B^{(2)}[3])$ will easily have a $K_{h,h}$-factor. With ϵ' small enough, we can guarantee that at most $(\epsilon')^{1/3}t_1$ vertices in $(B^{(1)}[2], B^{(3)}[2])$ and $(B^{(2)}[1], B_3[1])$ were moved. Applying Lemmas 3.8 and 3.9, and the fact that no pair other than $(B^{(1)}, B^{(2)})$ can be ϵ'-approximately $\Theta_{2\times2}(t_1)$, we conclude that the pairs $(B^{(1)}[2], B^{(3)}[2])$ and $(B^{(2)}[1], B_3[1])$ can be completed to $K_{h,h}$-factors.

Case 3: Exactly two pairs are $\Theta_{2\times2}(t_1)$, which do not coincide

Let the pairs in question be $(B^{(1)}, B^{(2)})$ and $(B^{(2)}, B^{(3)})$. Let the dense pairs in the subgraph induced by $(B^{(1)}, B^{(2)})$ be $(\widetilde{B}^{(1)}, \widetilde{B}^{(2)})$ and $(\widehat{B}^{(1)}, \widehat{B}^{(2)})$. Let the dense pairs in $(B^{(2)}, B^{(3)})$ be $(\mathring{B}^{(2)}, \mathring{B}^{(3)})$ and $(\ddot{B}^{(2)}, \ddot{B}^{(3)})$. Moreover, since the pairs fail to coincide, we can conclude that the intersection of the typical vertices of $\widetilde{B}^{(2)}$ with the typical vertices of each of $\mathring{B}^{(2)}$ and $\ddot{B}^{(2)}$ is at least $(\epsilon')^{1/4}t_1$ and similarly for $\widehat{B}^{(2)}$.

Once again, we randomly partition the vertices in $B^{(1)}$, $B^{(2)}$ and $B^{(3)}$ and move vertices so as to ensure that the reserved vertices and the vertices exchanged for them are placed into vertex-disjoint copies of $K_{h,h}$. Our concern at this point is the vertices in $B^{(2)}$.

Consider the vertices in $(B^{(1)}[3], B^{(2)}[3])$. Approximately half are typical vertices of $\widetilde{B}^{(2)}$ and approximately half are typical vertices of $\widehat{B}^{(2)}$. Take each non-typical vertex in $B^{(1)}[3]$ and in $B^{(2)}[3]$, match them with a copy of $K_{h,h}$ in the pair $(B^{(1)}[3], B^{(2)}[3])$ and remove them. Do the

same for vertices in $B^{(2)}[1]$ that are not typical in $\mathring{B}^{(2)}$ or $\ddot{B}^{(2)}$ and in $B^{(3)}[1]$ that are not typical in $\mathring{B}^{(3)}$ or $\ddot{B}^{(3)}$. Remove those copies of $K_{h,h}$ also.

Observe that there are at least $\epsilon^{1/4}t_1/4$ vertices in each intersection of $\widetilde{B}^{(2)}$ or $\widehat{B}^{(2)}$ with $\mathring{B}^{(2)}$ or $\ddot{B}^{(2)}$ and with $B^{(2)}[3]$ or $B^{(2)}[1]$.

First, move a vertices from $\widetilde{B}^{(2)} \cap \mathring{B}^{(2)} \cap B^{(2)}[3]$ to $\widetilde{B}^{(2)} \cap \mathring{B}^{(2)} \cap B^{(2)}[1]$ to make $|\widetilde{B}^{(2)} \cap B^{(2)}[3]|$ divisible by h. Second, move $a+b$ vertices from $\widehat{B}^{(2)} \cap \mathring{B}^{(2)} \cap B^{(2)}[1]$ to $\widehat{B}^{(2)} \cap \mathring{B}^{(2)} \cap B^{(2)}[3]$ to make $|\mathring{B}^{(2)} \cap B^{(2)}[1]|$ divisible by h. Third, move $a+b+c$ vertices from $\widehat{B}^{(2)} \cap \ddot{B}^{(2)} \cap B^{(2)}[3]$ to $\widehat{B}^{(2)} \cap \ddot{B}^{(2)} \cap B^{(2)}[1]$. This will make both $|\widehat{B}^{(2)} \cap B^{(2)}[3]|$ and $|\ddot{B}^{(2)} \cap B^{(2)}[1]|$ divisible by h.

Here a, b and c are the remainders of $|\widetilde{B}^{(2)} \cap B^{(2)}[3]|$, $|\mathring{B}^{(2)} \cap B^{(2)}[1]|$ and $|\widehat{B}^{(2)} \cap B^{(2)}[3]|$, respectively, when each is divided by h. Observe that both $|\widetilde{B}^{(2)} \cap B^{(2)}[3]| + |\widehat{B}^{(2)} \cap B^{(2)}[3]|$ and $|\mathring{B}^{(2)} \cap B^{(2)}[1]| + |\ddot{B}^{(2)} \cap B^{(2)}[1]|$ are divisible by h.

Finally, we exchange vertices in $\widetilde{B}^{(1)} \cap B^{(1)}[3]$ with those in $\widetilde{B}^{(1)} \cap B^{(1)}[2]$ so that $|\widetilde{B}^{(1)} \cap B^{(1)}[3]| = |\widetilde{B}^{(2)} \cap B^{(2)}[3]|$ and similarly for $\widehat{B}^{(2)}$. Also, exchange vertices in $\mathring{B}^{(3)} \cap B^{(3)}[1]$ with those in $\mathring{B}^{(3)} \cap B^{(3)}[2]$ so that $|\mathring{B}^{(3)} \cap B^{(3)}[1]| = |\mathring{B}^{(2)} \cap B^{(2)}[1]|$ and similarly for $\ddot{B}^{(2)}$.

Then, in $(\widetilde{B}^{(1)} \cap B^{(1)}[3], \widetilde{B}^{(2)} \cap B^{(2)}[3])$, first greedily place each moved vertex into copies of $K_{h,h}$ and then finish the factor via Proposition 3.7(1). Do the same for $\left(\widehat{B}^{(1)} \cap B^{(1)}[3], \widehat{B}^{(2)} \cap B^{(2)}[3] \right)$, $\left(\mathring{B}^{(2)} \cap B^{(2)}[1], \mathring{B}^{(3)} \cap B^{(3)}[1] \right)$ and $\left(\ddot{B}^{(2)} \cap B^{(2)}[1], \ddot{B}^{(3)} \cap B^{(3)}[1] \right)$.

Finally, we can complete the factor of $(B^{(1)}[2], B^{(3)}[2])$ because if it is not possible, Lemmas 3.8 and 3.9 would require $(B^{(1)}, B^{(3)})$ to be approximately $\Theta_{2\times2}(t_1)$, excluded by this case.

Case 4: Three pairs are $\Theta_{2\times2}(t_1)$, none of which coincide

Let the dense pairs in $(B^{(1)}, B^{(2)})$ be $(\widetilde{B}^{(1)}, \widetilde{B}^{(2)})$ and $(\widehat{B}^{(1)}, \widehat{B}^{(2)})$. Let the dense pairs in $(B^{(2)}, B^{(3)})$ be $(\mathring{B}^{(2)}, \mathring{B}^{(3)})$ and $(\ddot{B}^{(2)}, \ddot{B}^{(3)})$. Let the dense pairs in $(B^{(1)}, B^{(3)})$ be $(B^{(1)}_\sharp, B^{(3)}_\sharp)$ and $(B^{(1)}_\flat, B^{(3)}_\flat)$. Moreover, since the pairs fail to coincide, we can conclude that the intersection of the typical vertices of one set of sparse pairs with the typical vertices of another is at least $(\epsilon')^{1/4}t_1$.

Partition $B^{(1)}$, $B^{(2)}$ and $B^{(3)}$ into appropriately-sized sets as before, uniformly at random. The degree conditions hold with high probability as before. Take non-typical vertices and complete them greedily to place them in vertex-disjoint copies of $K_{h,h}$ within each of the pairs $(B^{(1)}[3], B^{(2)}[3])$, $(B^{(2)}[1], B^{(3)}[1])$ and $(B^{(1)}[2], B^{(3)}[2])$. Remove these copies of $K_{h,h}$ from the graph.

Let M be the largest multiple of h less than or equal to the size of the intersection of what remains of any sparse set (*i.e.*, $\widetilde{B}^{(i)}, \widehat{B}^{(i)}, \mathring{B}^{(i)}, \ddot{B}^{(i)}, B^{(i)}_\sharp, B^{(i)}_\flat$) with a set of the form $B^{(i)}[k]$.

We can move vertices as in Case 3 by letting $a = |\widetilde{B}^{(2)} \cap B^{(2)}[3]| - M$, $b = |\mathring{B}^{(2)} \cap B^{(2)}[1]| - M$ and $c = |\widehat{B}^{(2)} \cap B^{(2)}[3]| + M - t_3$, which is also equal to $t_1 - M - a - b - |\ddot{B}^{(2)} \cap B^{(2)}[1]|$. We can perform similar operations to guarantee that, among the vertices that remain in the graph, that

$$ M \;=\; \left| \widetilde{B}^{(1)} \cap B^{(1)}[3] \right| = \left| \widetilde{B}^{(2)} \cap B^{(2)}[3] \right| = \left| \mathring{B}^{(2)} \cap B^{(2)}[1] \right| = \left| \mathring{B}^{(3)} \cap B^{(3)}[1] \right| $$

$$= \left| B_\sharp^{(1)} \cap B^{(1)}[2] \right| = \left| B_\sharp^{(3)} \cap B^{(3)}[2] \right|$$

The fact that the pairs do not coincide ensures that there are enough vertices to make these moves.

Place the moved vertices into vertex-disjoint copies of $K_{h,h}$ and finish the factor via Proposition 3.7(1). $\qquad\square$

Proof of Proposition 3.7.

(1) This is found by arbitrarily placing vertices from the same part into clusters of size h. Construct an auxiliary graph G' on the clusters where two are adjacent if and only if they form a $K_{h,h}$ in G. Each cluster in G' is adjacent to at least half of the M/h clusters in the other part. Using König-Hall, we find a matching in G', producing a $K_{h,h}$-factor.

(2) The idea is the same as above – place vertices into clusters of size h – and use the tripartite version of Proposition 1.3 in [14] as a generalization of König-Hall.

$\qquad\square$

4 Concluding Remarks

- We are not sure whether the upper or lower bounds of Theorem 1.3 are correct. Since the very extreme case is so specific, it may be easier to improve the bounds for the case where N is an odd multiple of $3h$, but we were unable to do so.

- In [15], the minimum degree condition $\bar{\delta} \geq (2/3 + o(1))N$ for tiling $K_{h,h,h}$ was already established. The coefficient $2/3$ may not, however, be best possible for other 3-colorable graphs, e.g., $K_{1,2,3}$. In fact, Kühn and Osthus [11] gave a minimum degree condition for tiling a general (instead of 3-partite) graph with certain 3-colorable H's. Instead of the Alon-Yuster [1, 2] value of $1 - 1/\chi(H)$, the coefficient is $1 - 1/\chi_{cr}(H)$, where $\chi_{cr}(H)$ is the so-called critical chromatic number.

  Hladký and Schacht [8] recently established the coefficient for $K_{s,t}$ factors in bipartite graphs for $s < t$. It would be interesting to see if something similar holds for tripartite tiling.

- The general case for tiling an r-partite graph with just copies of K_r is unknown except in the quadripartite case [14] although an approximate version for larger r has been established by Csaba and Mydlarz [5].

5 Acknowledgments

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